Homological Algebra
Lecture 3

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Exactness in Abelian Categories

Suppose $\mathcal{A}$ is an abelian category. We now try to formulate what it means for a sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \tag{1}$$

to be exact. First of all we require that $gf = 0$. When it is, f factors through the kernel of g:

$$X \to \mathrm{Ker}(g) \to Y$$

But the composite

$$\mathrm{Ker}(f) \to X \to \mathrm{Ker}(g) \to Y$$

is 0 and $\mathrm{Ker}(g) \to Y$ is a monomorphism, so

$$\mathrm{Ker}(f) \to X \to \mathrm{Ker}(g)$$

is 0. Therefore f factors

$$X \to \mathrm{Im}(f) \to \mathrm{Ker}(g) \to Y.$$

Since $\mathrm{Im}(f) \to Y$ is a monomorphism,

$$\mathrm{Im}(f) \to \mathrm{Ker}(g)$$

is a monomorphism as well.

Definition

A sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is exact if $gf = 0$ and the canonical monomorphism

$$\mathrm{Im}(f) \to \mathrm{Ker}(g)$$

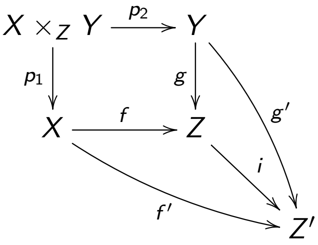
is an isomorphism.

We need the following lemma for the next proposition:

Lemma

Suppose $f : X \to Z$ and $g : Y \to Z$ are morphisms in a category $\mathcal{C}$ which has fibered products, and suppose $i : Z \to Z'$ is a monomorphism. Set $f' = if : X \to Z'$ and $g' = ig : Y \to Z'$. The canonical morphism $X \times_Z Y \to X \times_{Z'} Y$ is an isomorphism.

Proof: The canonical morphism comes from applying the universal property of the fibered product to the diagram



The universal property of $X \times_Z Y$ is that the set of morphisms $T \to X \times_Z Y$ is in a functorial bijection with the set of pairs of morphisms $a : T \to X$ and $b : T \to Y$ such that $fa = gb$. Since i is a monomorphism, $fa = gb$ if and only if $ifa = igb$, i.e. $f'a = g'b$. Thus $X \times_Z Y$ and $X \times_{Z'} Y$ solve the same universal problem, i.e. represent the same functor. ∎

Alternate method: observe that the lemma holds in the category of sets and then use Yoneda. Exercise: this is really the same as the first argument.

Returning to the subject of exactness, it can be rather difficult to show that a sequence $X \to Y \to Z$ is exact using the definition. Therefore the following proposition is useful:

Proposition

For any diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

the following are equivalent:

1. *The diagram is an exact sequence.*
2. $gf = 0$, *and for any morphism* $h : T \to Y$ *such that* $gh = 0$ *the projection* $p_2 : X \times_Y T \to T$ *is an epimorphism.*
3. $gf = 0$, *and for any morphism* $h : T \to Y$ *such that* $gh = 0$ *there is a commutative diagram*

$$\begin{array}{ccc} T' & \twoheadrightarrow & T \\ \downarrow & & \downarrow h \\ X & \xrightarrow{f} & Y \end{array} \tag{2}$$

in which $T' \to T$ *is an epimorphism.*

Proof: (1) implies (2): Since $gh = 0$, $h : T \to Y$ factors $T \to \operatorname{Ker}(g) \to Y$. Since $\operatorname{Im}(f) \to \operatorname{Ker}(g)$ is an isomorphism, h factors $T \to \operatorname{Im}(f) \to Y$. Consider now the diagram

$$
\begin{array}{ccccc}
X \times_{\operatorname{Im}(f)} T & \longrightarrow & T & & \\
\downarrow & & \downarrow & & \\
X & \twoheadrightarrow & \operatorname{Im}(f) & \hookrightarrow & Y
\end{array}
$$

Since $X \to \operatorname{Im}(f)$ is an epimorphism and $\mathcal{A}$ is abelian, $X \times_{\operatorname{Im}(f)} T \to T$ is an epimorphism. On the other hand since $\operatorname{Im}(f) \to Y$ is a monomorphism then natural morphism $X \times_{\operatorname{Im}(f)} T \to X \times_Y T$ is an isomorphism. Therefore $p_2 : X \times_Y T \to T$ is an epimorphism.

(2) implies (3): Take $T' \to T$ to be $p_2 : X \times_Y T \to T$.

(3) implies (1): We know that $\mathrm{Im}(f) \to \mathrm{Ker}(g)$ is a monomorphism, so it suffices to show that it is an epimorphism. Take $T = \mathrm{Ker}(g)$, let $h : T \to Y$ be the canonical monomorphism and let $\ell : T' \to T = \mathrm{Ker}(g)$ be an epimorphism making commutative the diagram in (3). Since ℓ is the composite $T' \to X \to \mathrm{Im}(f) \to \mathrm{Ker}(g)$, the morphism $\mathrm{Im}(f) \to \mathrm{Ker}(g)$ is an epimorphism by Chevalley's lemma. ∎

We say, more generally that a sequence (finite or infinite) of composable morphisms

$$\cdots \xrightarrow{f_{n-2}} X_{n-1} \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} \cdots$$

is an *exact sequence* if $f_n f_{n-1} = 0$ for all n and

$$X_{n-1} \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1}$$

is exact for all n, i.e. if the canonical morphism $\mathrm{Im}(f_{n-1}) \to \mathrm{Ker}(f_n)$ is an isomorphism for all n.

For example if $A \to B$ is the kernel of $B \to C$ the sequence

$$0 \to A \to B \to C$$

is exact. First, the composites $0 \to A \to B$ and $A \to B \to C$ are zero. Second, if $T \to A$ is such that $T \to A \to B$ is zero then T is zero since $A \to B$ is a monomorphism, and then the diagram

$$\begin{array}{ccc} 0 & \longrightarrow & T \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & A \end{array}$$

satisfies condition (3) of the last proposition. Suppose $T \to B$ is such that $T \to B \to C$ is zero. Since $A \to B$ is a kernel of $B \to C$, $T \to B$ arises by composition from a unique morphism $T \to A$ and

$$\begin{array}{ccc} T & \xrightarrow{1} & T \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

again satisfies condition (3). Similarly if $B \to C$ is a cokernel of $A \to B$ the sequence

$$A \to B \to C \to 0$$

is an exact sequence.

We now have all we need to prove that the snake lemma holds in any abelian category. Recall the picture:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & A' & \xrightarrow[i']{} & B' & \xrightarrow[p']{} & C' & \longrightarrow & 0 \end{array}$$

and the assertion is that there is an exact sequence

$$0 \to \mathrm{Ker}(a) \to \mathrm{Ker}(b) \to \mathrm{Ker}(c) \xrightarrow{\partial} \mathrm{Coker}(a) \to \mathrm{Coker}(b) \to \mathrm{Coker}(c) \to 0.$$

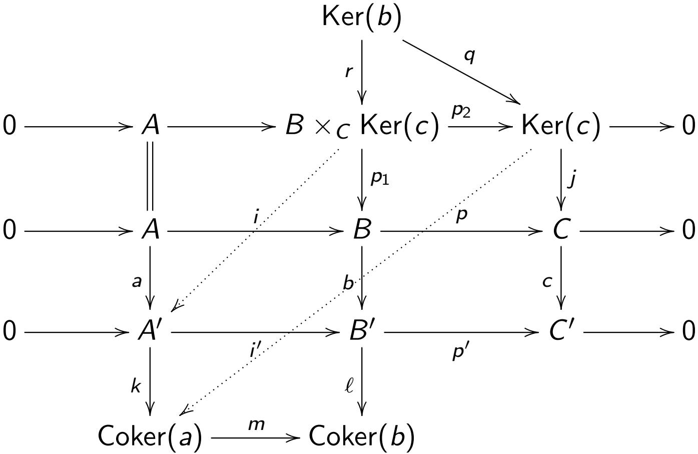
We have already shown that the sequences

$$0 \to \mathrm{Ker}(a) \to \mathrm{Ker}(b) \to \mathrm{Ker}(c)$$
$$\mathrm{Coker}(a) \to \mathrm{Coker}(b) \to \mathrm{Coker}(c) \to 0$$

are exact. It remains to construct ∂ and show that

$$\mathrm{Ker}(b) \to \mathrm{Ker}(c) \xrightarrow{\partial} \mathrm{Coker}(a) \to \mathrm{Coker}(b)$$

is exact. The following arguments all refer to the diagram



in which the rows are exact and r is induced by $\mathrm{Ker}(b) \to B$ and $\mathrm{Ker}(b) \to \mathrm{Ker}(c)$.

(1) To define ∂ we first note that

$$p' b p_1 = c j p_2 = 0 \quad \text{since} \quad cj = 0.$$

Therefore thre is a unique morphism $f : B \times_C \mathrm{Ker}(c) \to A'$ (the upper dotted one in the diagram) such that

$$i' f = b p_1.$$

Since $A \to B \times_C \mathrm{Ker}(c)$ is the kernel of $p_2 : B \times_C \mathrm{Ker}(c) \to \mathrm{Ker}(c)$ and $A \to A' \to \mathrm{Coker}(a)$ is zero, we can define $\partial : \mathrm{Ker}(c) \to \mathrm{Coker}(a)$ as the unique morphism such that

$$\partial p_2 = kf.$$

(2) Check $\partial q = 0$: by construction $\partial q = \partial p_2 r = kfr$, but $i' fr = b p_1 r = 0$ since the latter morphism $\mathrm{Ker}(b) \to B \to B'$. Since i' is a monomorphism, $fr = 0$ and therefore $\partial q = kfr = 0$.

(3) Check $m\partial = 0$: by construction

$$m\partial p_2 = mkf = \ell i' f = \ell b p_1 = 0$$

since already $\ell b = 0$. Since p_2 is an epimorhism, $m\partial = 0$.

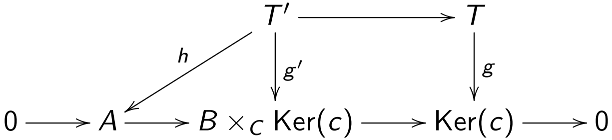
(4) Check that

$$\mathrm{Ker}(b) \xrightarrow{b} \mathrm{Ker}(c) \xrightarrow{\partial} \mathrm{Coker}(a)$$

is exact: Suppose $g : T \to \mathrm{Ker}(c)$ is such that $\partial g = 0$. There is a commutative square

$$\begin{array}{ccc} T' & \longrightarrow & T \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ B \times_C \mathrm{Ker}(c) & \xrightarrow[p_2]{} & \mathrm{Ker}(c) \end{array}$$

in which $T' \to T$ is an epimorphism (e.g. take T' to be the fibered product). From $\partial g = 0$ we deduce, from the large diagram that $kfg' = 0$, and since k is the cokernel of $a : A \to A'$ we get a morphism $h : T' \to A$. The diagram



is commutative, so

$$T' \xrightarrow{g'} B \times_C \operatorname{Ker}(c) \to \operatorname{Ker}(c)$$

is zero, i.e. g' factors through a morphism

$$T' \to \operatorname{Ker}(B \times_C \operatorname{Ker}(c) \to \operatorname{Ker}(c)).$$

But the target here is isomorphic to $\operatorname{Ker}(b)$, so we have a commutative square

$$\begin{array}{ccc} T' & \twoheadrightarrow & T \\ \downarrow & & \downarrow \\ \mathrm{Ker}(b) & \longrightarrow & \mathrm{Ker}(c) \end{array}$$

showing that $\mathrm{Ker}(b) \to \mathrm{Ker}(c) \xrightarrow{\partial} \mathrm{Coker}(a)$ is exact.

(5) Check that

$$\mathrm{Ker}(c) \xrightarrow{\partial} \mathrm{Coker}(a) \to \mathrm{Coker}(b)$$

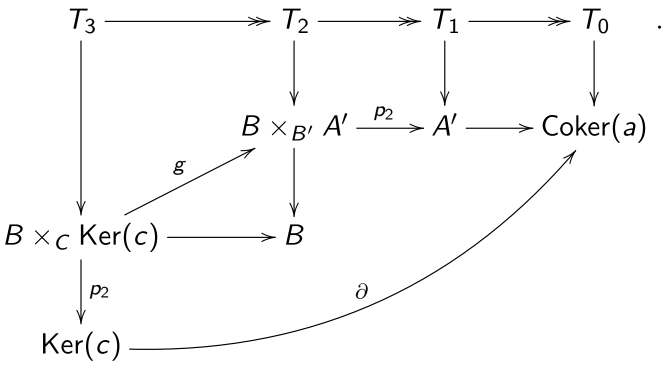
is exact: Suppose $T_0 \to \mathrm{Coker}(a)$ is a morphism such that the composition $T_0 \to \mathrm{Coker}(a) \to \mathrm{Coker}(b)$ is zero. Successive applications of the proposition lead to a diagram

$$
\begin{array}{ccccccccc}
T_3 & \twoheadrightarrow & T_2 & \twoheadrightarrow & T_1 & \twoheadrightarrow & T_0 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\downarrow & & \downarrow & & A' & \rightarrow & \operatorname{Coker}(a) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\downarrow & & B & \rightarrow & B' & \rightarrow & \operatorname{Coker}(b) & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
\operatorname{Ker}(c) & \rightarrow & C & \rightarrow & C' & & & &
\end{array}
\quad .
$$

We then observe that the morphisms $T_3 \to \operatorname{Ker}(c)$ and $T_2 \to B$ factor through fibered products, leading to a diagram

$$
\begin{array}{ccccccc}
T_3 & \twoheadrightarrow & T_2 & \twoheadrightarrow & T_1 & \twoheadrightarrow & T_0 \\
\Big\downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & B \times_{B'} A' & \longrightarrow & A' & \longrightarrow & \mathrm{Coker}(a) \\
\Big\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B \times_C \mathrm{Ker}(c) & \longrightarrow & B & \longrightarrow & B' & \longrightarrow & \mathrm{Coker}(b) \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Ker}(c) & \longrightarrow & C & \longrightarrow & C' & &
\end{array}
\quad .
$$

Observe finally that the morphism $f : B \times_C \mathrm{Ker}(c) \to A'$ figuring in the construction of ∂ factors through a morphism $g : B \times_C \mathrm{Ker}(c) \to B \times_{B'} A'$. We now have a commutative diagram



in which $f = p_2 g$. Collapsing the top and left sides produces a commutative square

$$\begin{array}{ccc} T_3 & \twoheadrightarrow & T_0 \\ \downarrow & & \downarrow \\ \mathrm{Ker}(c) & \xrightarrow{\partial} & \mathrm{Coker}(a) \end{array}$$

which shows that

$$\mathrm{Ker}(c) \xrightarrow{\partial} \mathrm{Coker}(a) \to \mathrm{Coker}(b) \tag{3}$$

is exact.

Note the similarity of the last argument with the argument we would make if we to show that (3) is exact if we in some category of modules and chasing elements through a diagram: in effect, the role of $T \to Y$ and $T' \to X$ in the proposition play the role of “elements” of Y and X.

In fact however one can usually prove theorems of this sort by *pretending* we are in a module category and chasing elements through diagrams. The justification for this is that usually the diagram you are looking at has a *set* of objects, in which case you can invoke the following theorem, which I will not prove; there is a sketch of a proof in Weibel's book, section 1.6.

Theorem (Freyd-Mitchell)

If $\mathcal{A}$ is a small abelian category there is a ring R and a fully faithful functor $\mathcal{A} \to \mathbf{Mod}_R$.

From this one can simply *deduce* the snake lemma for an arbitrary category $\mathcal{A}$ from its truth in any category of modules; one need only replace $\mathcal{A}$ by the smallest full abelian subcategory containing the six objects A, $B, \ldots, C'$ and the morphisms between them. Of course one needs to prove that such a "smallest abelian full subcategory" containing a given set of objects and morphisms exists, which I will leave as an exercise.

No one will ever blame you for checking assertions like the snake lemma by using the Freyd-Mitchell theorem, but everyone should see a proof in the abstract setting done at least once. It is important to keep in mind that not every abelian category has an "obvious" interpretation as a module category, the main examples being sheaf categories. There are in fact some monumentally *wierd* abelian categories out there, which you will see later if you are "lucky."