Homological Algebra
Lecture 4

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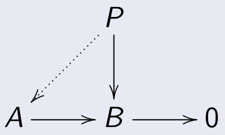
Summer 2021

Projective and Injective objects

The definition of a projective or injective module involves nothing but general categorical notions (objects, morphisms) and the notion of an exact sequence. It therefore makes sense in an arbitrary abelian category. Let $\mathcal{A}$ be an abelian category

Definition

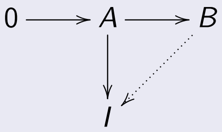
An object P of $\mathcal{A}$ is *projective* if for every diagram



in which the bottom row is exact, the dotted arrow can be filled in.

Definition

An object I of $\mathcal{A}$ is *injective* if for every diagram



in which the top row is exact, the dotted arrow can be filled in.

The two definitions are clearly dual: an object M of $\mathcal{A}$ is projective (resp. injective) if and only it is injective (resp. projective) as an object of $\mathcal{A}^{\mathrm{op}}$.

The definitions can be rephrased as follows: P is projective if and only if

$$\mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(P, A)$$

is surjective for every epimorphism $A \to B$.

Dually, I is injective if adn only if

$$\mathrm{Hom}_R(B, I) \to \mathrm{Hom}_R(A, I)$$

is surjective for every monomorphism $A \to B$.

We say that $\mathcal{A}$ *has enough projectives (resp. injectives)* if for every object M of $\mathcal{A}$ there is a epimorphism $P \to M$ with P projective (resp. a monomorphism $M \to I$ with I injective).

(Enough for *what*, you ask?)

Again it is clear that $\mathcal{A}$ has enough projectives (resp. injectives) if and only if $\mathcal{A}^{\mathrm{op}}$ has enough injectives (resp. projectives). Nonetheless there is a fundamental asymmetry in nature manifested by the fact that many interesting abelian categories have enough injectives but not enough projectives. Therefore most results will be stated in terms of injectives, even though the corresponding statements for projectives are true (and follow by duality).

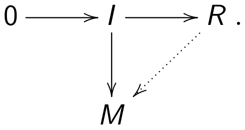
If R is a ring, the category $\mathbf{Mod}_R$ of left R-modules has enough projectives. In fact any free R-module is projective, and for any R-module M there is a surjective homomorphism $F \to M$ for some free R-module F (e.g. the free R-module with basis consisting of the elements of M).

The category $\mathbf{Mod}_R$ also has enough injectives. This is a little harder since it is not so obvious how to construct injective modules. We first recall the following criterion:

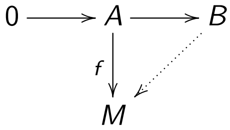
Theorem (Baer)

Suppose R is a ring with identity. A left R-module M is injective if and only if for every left ideal $I \subseteq R$, an R-module homomorphism $I \to M$ extends to an R-module homomorphism $R \to M$.

Proof: That the condition is necessary follows from applying the definition to the diagram



To show it is sufficient we consider the diagram



Let $\mathcal{S}$ be the set of pairs (C, g) with C a submodule $A \subseteq C \subseteq B$ and

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \longrightarrow & C \\ & & {\scriptstyle f}\downarrow & \swarrow {\scriptstyle g} & \\ & & M & & \end{array}$$

commutes. We introduce a partial order on $\mathcal{S}$ by saying that $(C, g) \leq (C', g')$ if $C \subseteq C'$ and g' extends g. The set $\mathcal{S}$ is inductive for this order, so it has a maximal element (E, h) by Zorn. If $E \neq B$, pick $x \in B \setminus E$; the set

$$I = \{r \in R \,|\, rx \in E\}$$

is a left ideal of R. Since $r \mapsto h(rx)$ is an R-module homomorphism $I \to M$, by hypothesis there is an R-module homomorphism $\ell : R \to M$ such that $\ell(r) = h(rx)$ for all $r \in I$. Suppose e, $e' \in E$ and r, $r' \in R$. If

$$e' + r'x = e + rx$$

then $e' - e = (r - r')x \in E$ and thus $r - r' \in I$. Therefore

$$h(e') - h(e) = h(e' - e) = h((r - r')x) = \ell(r - r') = \ell(r) - \ell(r')$$

or

$$h(e') + \ell(r') = h(e) + \ell(r).$$

Thus if $F = E + Rx$ we can extend $h : E \to M$ to $h' : F \to M$ by setting

$$h'(e + rx) = h(e) + \ell(r).$$

This contradicts the maximality of (E, h), so $E = B$. ■

Corollary

If R is a PID, an R-module M is injective if and only if it is divisible.

Proof: Suppose M is divisible and $I = (a)$ is an ideal of R. If $a = 0$, $f = 0$ extends to the zero homomorphism $R \to M$. Otherwise we can pick $m \in M$ such that $am = f(a)$; an extension is $r \mapsto rm$. The converse is left as an exercise. ■

Suppose now R is a commutative ring and A is an R-algebra. For any R-module N, $\mathrm{Hom}_R(A, N)$ has a left A-module structure coming from the right A-module structure of A inside the Hom: for $a \in A$ and R-linear $f : A \to N$, (af) sends $x \in A$ to $f(ax) \in N$. For any A-module M there is an isomorphism

$$\mathrm{Hom}_R(M, N) \xrightarrow{\sim} \mathrm{Hom}_A(M, \mathrm{Hom}_R(A, N)) \tag{1}$$

functorial in both M and N. To an R-linear $f : M \to N$ it assigns the A-linear map sending m to $[a \mapsto f(am)]$. You should check the asserted linearities! The inverse map assigns to an A-linear $g : M \to \mathrm{Hom}_R(A, N)$ the map R-linear map $M \to N$ which evaluates g on $m \in M$ to obtain an R-linear $g(m) : A \to N$ then then evaluates $g(m)$ on $1 \in A$. Again, you should check the asserted linearities and that this is an inverse to (1).

We now take $R = \mathbb{Z}$.

Lemma

For any ring R and divisible abelian group D, $\mathrm{Hom}_{\mathbb{Z}}(R, D)$ is an injective R-module.

Proof: Suppose $M \to N$ is a monomorphism: we must show that

$$\mathrm{Hom}_R(N, \mathrm{Hom}_{\mathbb{Z}}(R, D)) \to \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R, D))$$

is surjective. The functorial isomorphism (1) shows that this is equivalent to the surjectivity of

$$\mathrm{Hom}_{\mathbb{Z}}(N, D) \to \mathrm{Hom}_{\mathbb{Z}}(M, D).$$

Since D is divisible it is an injective $\mathbb{Z}$-module, so this is surjective. ∎

Remark: The functorial isomorphism (1) shows that the forgetful functor $\mathbf{Mod}_A \to \mathbf{Mod}_R$ has a right adjoint $F : \mathbf{Mod}_R \to \mathbf{Mod}_A$, and F has a left adjoint (the forgetful functor). If $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $F : \mathcal{A} \to \mathcal{B}$ is a functor with a left adjoint, the above argument shows that if I is an injective object of $\mathcal{A}$ then $F(I)$ is an injective object of $\mathcal{B}$.

The category of abelian groups has enough injectives:

Lemma

For any abelian group A there is an injective homomorphism $A \to D$ with D divisible.

Proof: Let $D = (\mathbb{Q}/\mathbb{Z})^I$ where $I = \mathrm{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z})$ i.e. D is the product of copies of $\mathbb{Q}/\mathbb{Z}$ indexed by the set $\mathrm{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z})$. A product of divisible groups is divisible, so D is divisible.

Let $e : A \to D$ be the map which to $a \in A$ assigns the I-tuple $(f(a))_{f \in I}$. If $e(a) = e(b)$ then $f(a) = f(b)$ for *all* $f : A \to \mathbb{Q}/\mathbb{Z}$, so e will be injective if for every nonzero $a \in A$ there is an $f : A \to \mathbb{Q}/\mathbb{Z}$ such that $f(a) \neq 0$. Since $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module it suffices to find a nonzero homomorphism $\mathbb{Z}a \to \mathbb{Q}/\mathbb{Z}$. In fact if a has additive order n we can send $a \mapsto 1/n$; if a has infinite order we can send it to any nonzero element of $\mathbb{Q}/\mathbb{Z}$. In either case the result is a nonzero homomorphism $\mathbb{Z}a \to \mathbb{Q}/\mathbb{Z}$. ■

Theorem

For any ring with identity R, the category $\mathbf{Mod}_R$ has enough injectives.

Proof: Let M be an R-module. By the previous lemma there is an injective homomorphism $M \to D$ of abelian groups with D divisible. The composite homomorphism

$$M \xrightarrow{\sim} \mathrm{Hom}_R(R, M) \to \mathrm{Hom}_{\mathbb{Z}}(R, M) \to \mathrm{Hom}_{\mathbb{Z}}(R, D)$$

is injective and R-linear. The penultimate lemma says that $\mathrm{Hom}_{\mathbb{Z}}(R, D)$ is an injective R-module, so we are done. ■

There are not many *general* results on the existence of injectives in abelian categories, but the following results are often useful.

Lemma

If $\mathcal{A}$ is an abelian category and

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0 \qquad (2)$$

is an exact sequence, the following are equivalent:

1. *there is a morphism $s : C \to B$ such that $ps = 1_C$;*
2. *there is a morphism $t : B \to A$ such that $ti = 1_A$;*
3. *there is a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C & \longrightarrow & 0 \\
 & & \| & & \downarrow u & & \| & & \\
0 & \longrightarrow & A & \xrightarrow{i_1} & A \oplus C & \xrightarrow{p_2} & C & \longrightarrow & 0
\end{array}$$

This is proven in the same way as in module categories, so I will just sketch the proof.

If (3) holds then u is an isomorphism (snake lemma) and then $s = u^{-1}i_2$ and $t = p_1 u$ satisfy the conditions of (1) and (2). If (1) holds, $psp = p$ and thus $p(1 - sp) = 0$ and there is a unique $j : B \to A$ such that $i_1 j = 1 - sp$. We may then take $u = (j, sp)$. The argument that (2) implies (3) is similar (in fact dual).

Definition

An exact sequence in an abelian category

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

is *split*, or *splits* if the equivalent conditions of the lemma hold for it.

The morphisms s and t are called *splittings* of the exact sequence.

Proposition

An object P is projective if and only if any exact sequence

$$0 \to A \to B \to P \to 0$$

splits.

Proof: if P is projective, a splitting $s : P \to B$ arises by applying the definition of projective:

$$\begin{array}{ccccccccc} & & & & & & P & & \\ & & & & & \overset{s}{\swarrow} & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & P & \longrightarrow & 0 \end{array} .$$

Suppose conversely that any exact sequence as in the proposition splits and the diagram

$$\begin{array}{ccccccccc} & & & & & & P & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

is given. By an earlier proposition the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B \times_C P & \longrightarrow & P & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

has exact rows. By assumption the top row has a splitting $s : P \to B \times_C P$ and then it is easily checked that the morphism $p_1 s : P \to B$ lifts $P \to C$. ■

The dual argument proves:

Proposition

An object I is injective if and only if any exact sequence

$$0 \to I \to B \to C \to 0$$

splits.

Either the last two propositions, or a direct appeal to the definitions show the following:

Proposition

If $P \oplus Q$ is projective then so are P and Q. If $I \times J$ is injective so are I and J.

Of course $I \times J$ is isomorphic to $I \oplus J$ but the universal property used in the proof is that of the product, not the coproduct. The converse has a stronger version:

Proposition

If $\{P_\alpha\}_{\alpha \in S}$ (resp. $\{I_\alpha\}_{\alpha \in S}$) is a family of projectives (resp. injectives) then $\bigoplus_{\alpha \in S} P_\alpha$ is projective (resp. $\prod_{\alpha \in S} I_\alpha$ is injective).

Proof: exercise in using the definitions. ■

Note that a special case was used earlier: a product of divisible (i.e. injective) abelian groups is divisible.

In some cases a generalization of Baer's criterion is available.

Lemma

Suppose G is an object of an abelian category $\mathcal{A}$. The following are equivalent:

1. *For every monomorphism $i : A \to B$ that is not an isomorphism there is a morphism $G \to B$ that does not factor through i.*
2. *For every pair of distinct morphisms $f, g : A \to B$ there is a morphism $u : G \to A$ such that $fu \neq gu$.*

Proof: Suppose (1) holds and let $e : E \to A$ be an equalizer of f and g. Since $f \neq g$, e is a monomorphism that is not an isomorphism, so there is a morphism $G \to A$ that does not factor through e. This implies that $fu \neq gu$.

Suppose conversely (2) holds and let $i : A \to B$ be a monomorphism that is not an isomorphism. Let $p : B \to C$ be a cokernel of i; since i is not an isomorphism C is not zero, i.e. $p \neq 0$. Applying (2) with $f = p$ and $g = 0$ we find there is a morphism $u : G \to B$ such that $pu \neq 0$. Since i is the kernel of p this shows that p does not factor through i. ∎

Remarks: (i) The argument that (1) implies (2) works in a preabelian category. (ii) Condition (2) can be reformulated as sayin that if $f : A \to B$ is nonzero there is a $u : G \to A$ such that $fu \neq 0$, and this is the form that was actually used in the proof.

We say that an object G is a *generator* of $\mathcal{A}$ if it satisfies the equivalent conditions of the lemma. It is a *cogenerator* if it a generator in $\mathcal{A}^{\mathrm{op}}$, i.e. the conditions in the lemma hold with all the arrows reversed.

Example: In the category $\mathbf{Mod}_R$ of R-modules, the object R is a generator.

Let $\mathcal{C}$ be a category and recall that a subobject of an object M of $\mathcal{C}$ is a monomorphism $N \to M$. Subobjects $i : N \to M$, $i' : N' \to M$ are isomorphic if there is an isomorphism $j : N \xrightarrow{\sim} N'$ such that $ji' = i$. If $\mathcal{A}$ has a generator any object M of $\mathcal{A}$ has a *set* of isomorphism classes of subobjects, i.e. there is a set S of subobjects of M such that any subobject of M is isomorphic to some element of S. In fact any subobject $N \to M$ is determined up to isomorphism by the set of morphisms $G \to M$ that factor through N (exercise). A category with this property is said to be *well-powered*.

In order to state the analogue of Baer's criterion the following axioms first stated by Grothendieck will be useful. The first two simply define what it means for a preabelian category $\mathcal{A}$ to be abelian.

AB1 Any morphism in $\mathcal{A}$ has a kernel and cokernel.

AB2 For any morphism $f : A \to B$ the canonical morphism $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism.

AB3 Arbitrary coproducts in $\mathcal{A}$ are representable.

AB4 Axiom **AB3** holds, and if $\{u_i : A_i \to B_i\}$ is any set of monomorphisms,

$$\bigoplus_i u_i : \bigoplus_i A_i \to \bigoplus_i B_i$$

is a monomorphism.

AB5 Axiom **AB4** holds and filtered colimits commute with fibered products. In other words if A is an object of $\mathcal{A}$, I is a filtered category, $F : I \to \mathcal{A}_{/A}$ is a is a functor and $B \to A$ is a morphism, the canonical morphism

$$(\varinjlim_i F(i)) \times_A B \to \varinjlim_i (F(i) \times_A B)$$

is an isomorphism.

Theorem

Suppose $\mathcal{A}$ is an abelian category satisfying axiom **AB5**, *and $\mathcal{A}$ has a generator G. An object M of $\mathcal{A}$ is injective if and only if for every subobject $i : H \to G$ of G, a morphism $u : H \to M$ extends to a morphism $G \to M$.*

A proof can be found in Grothendieck's Tohoku paper and I will merely say that it basically follows the argument for Baer's criterion. The axiom **AB5** is used to execute the Zorn's lemma argument. From this he then deduces:

Theorem

Suppose $\mathcal{A}$ is an abelian category satisfying axiom **AB5**, *and $\mathcal{A}$ has a generator. Then $\mathcal{A}$ has enough injectives.*

The argument is similar to the one given for module categories.