Homological Algebra
Lecture 5

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A bit more about exactness

Recall that a diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in an abelian category is a complex if $gf = 0$. In this case f has a natural factorization

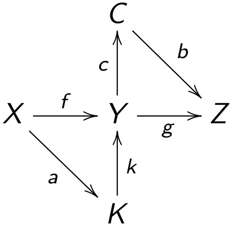
$$X \to \mathrm{Im}(f) \xrightarrow{h} \mathrm{Ker}(g) \to Y$$

and the complex is *exact* if h is an isomorphism. One might ask what the dual of this construction is. It turns out to be a factorization

$$Y \to \mathrm{Coker}(f) \xrightarrow{h'} \mathrm{Im}(g) \to Z$$

and h' is an epimorphism. It is important for what follows that $\mathrm{Coker}(h)$ and $\mathrm{Ker}(h')$ are isomorphic; in particular the complex is exact if and only if h' is an isomorphism. Let's prove this.

For simplicity we put $K = \mathrm{Ker}(g)$ and $C = \mathrm{Coker}(f)$. Then $gf = 0$ implies the existence of morphisms $a : X \to K$ and $b : C \to Z$ making



commutative. If we set $h = ck$ then $ha = cka = cf = 0$ and $bh = bck = gk = 0$. By the results of the previous slide there is a natural epimorphism $\mathrm{Coker}(a) \to \mathrm{Im}(h)$ and a natural monomorphism $\mathrm{Im}(h) \to \mathrm{Ker}(b)$. Combining these we get a morphism $\mathrm{Coker}(a) \to \mathrm{Ker}(b)$.

Lemma

The natural morphism $\mathrm{Coker}(a) \to \mathrm{Ker}(b)$ *is an isomorphism.*

We first prove this under the assumption that f is a monomorphism and g is an epimorphism, and then the general case will be reduced to this. In this situation f is a kernel of c and g is a cokernel of k.

Let $h = ck : K \to C$. I claim that a is a kernel of h and b is a cokernel of h; this will prove the lemma since the canonical factorization of h, which by definition is

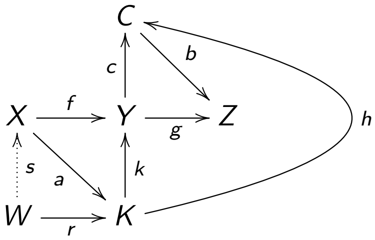
$$K \to \mathrm{Coim}(h) \to \mathrm{Im}(h) \to C$$

will in this case be

$$K \to \mathrm{Coker}(a) \to \mathrm{Ker}(b) \to C$$

and in an abelian category $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism.

To show that a is a kernel of h we must show (i) $ha = 0$ and (ii) if $r : W \to K$ is such that $hr = 0$ then r factors uniquely as $r = as$ for some $s : W \to X$.

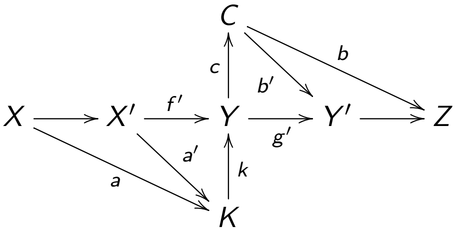


(i) $ha = cka = cf = 0$ since c is the cokernel of f.

(ii) Suppose $r : W \to K$ is such that $hr = 0$. Then $ckr = 0$, and since f is a kernel of c there is a (unique) morphism $s : W \to X$ such that $fs = kr$. Since $f = ka$, $kas = kr$ and therefore $as = r$ since k is a monomorphism (it's a kernel). Suppose $s' : W \to X$ is another morphism such that $as' = r$. Then $kas' = kr = kas$, which is $fs' = fs$, and since f is a monomorphism, $s' = s$.

The proof that b is a cokernel of h is dual to this.

Finally we reduce the general case to the one just proven. Let $X \to X' \xrightarrow{f'} Y$ and $Y \xrightarrow{g'} Y' \to Z$ be the canonical factorizations, i.e. $X' = \mathrm{Im}(f)$ and $Y' = \mathrm{Im}(g)$. Since $X \to X'$ is an epimorphism and $Y' \to Y$ is a monomorphism, $g'f' = 0$. Since $X \to X'$ is an epimorphism and $Y' \to Z$ is a monomorphism we can identify $\mathrm{Ker}(g) \simeq \mathrm{Ker}(g')$ and $\mathrm{Coker}(f) \simeq \mathrm{Coker}(f')$. The result is a diagram



in which f' is a monomorphism and g' is an epimorphism. We have already shown that the canonical $\mathrm{Coker}(a') \to \mathrm{Ker}(b')$ is an isomorphism, so it suffices to show that the canonical maps $\mathrm{Coker}(a) \to \mathrm{Coker}(a')$ and $\mathrm{Ker}(b') \to \mathrm{Ker}(b)$ are isomorphisms.

For any object T there is a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}(\mathrm{Coker}(a'), T) & \longrightarrow & \mathrm{Hom}(K, T) & \longrightarrow & \mathrm{Hom}(X', T) \\
 & & \downarrow & & \| & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}(\mathrm{Coker}(a), T) & \longrightarrow & \mathrm{Hom}(K, T) & \longrightarrow & \mathrm{Hom}(X, T)
\end{array}$$

with exact rows; the right vertical map is injective since $X \to X'$ is an epimorphism. Because it is injective, the snake lemma shows that the left vertical map is a bijection (if you like, replace the terms on the right by the image of $\mathrm{Hom}(K, T)$ in both). Therefore $\mathrm{Coker}(a) \to \mathrm{Coker}(a')$ is an isomorphism. The case of kernels is dual.

Let's now interpret this lemma. We have seen that the canonical factorization of $f : X \to Y$ itself factors

$$X \twoheadrightarrow \mathrm{Im}(f) \hookrightarrow \mathrm{Ker}(g) \hookrightarrow Y$$

and we define the *homology* of this complex to be

$$H(f, g) = \mathrm{Coker}(\mathrm{Im}(f) \to \mathrm{Ker}(g))$$

On the other hand the morphism a in the lemma is the composite $X \to \mathrm{Im}(f) \to \mathrm{Ker}(g)$, and since $X \to \mathrm{Im}(f)$ is an epimorphism, $X \to \mathrm{Ker}(g)$ and $\mathrm{Im}(f) \to \mathrm{Ker}(g)$ have the same cokernel (this was the argument of the last part of the lemma). Therefore

$$\mathrm{Coker}(a) \simeq H(f, g).$$

On the other hand the canonical factorization of g is $Y \to \mathrm{Coker}(f) \to Z$, which factors

$$Y \twoheadrightarrow \mathrm{Coker}(f) \twoheadrightarrow \mathrm{Im}(g) \hookrightarrow Z$$

(dual argument to the one before). But again, $c : \mathrm{Coker}(f) \to Z$ is the composite $\mathrm{Coker}(f) \to \mathrm{Im}(g) \to Z$, and the kernels of c and $\mathrm{Coker}(f) \to \mathrm{Im}(g)$ are isomorphic since $\mathrm{Im}(g) \to Z$ is a monomorphism. The lemma therefore says that

$$H(f, g) \simeq \mathrm{Ker}(c).$$

Recalling that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact if and only if $H(f, g)$ is zero, we have another criterion for the exactness of a complex:

Lemma

A complex $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact if and only if the canonical surjection $\mathrm{Coker}(f) \to \mathrm{Im}(g)$ is an isomorphism.

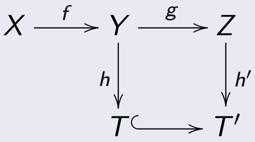
Recall also our earlier result that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact if and only if for any morphism $h : T \to Y$ such that $gh = 0$ there is a commutative diagram

$$\begin{array}{ccccc} T' & \twoheadrightarrow & T & & \\ \downarrow & & \downarrow{\scriptstyle h} & & \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z \end{array}$$

with $T' \to T$ an epimorphism; if it is exact, one such epimorphism is $p_2 : X \times_Y T \to T$.

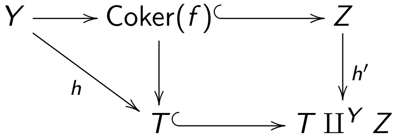
Proposition

A complex $X \xrightarrow{f} Y \xrightarrow{g} Z$ *is exact if and only if for every morphism* $h : Y \to T$ *such that* $hf = 0$ *there is a commutative diagram*



where $T \to T'$ *is a monomorphism.*

The proof is entirely dual to the earlier one so I will just indicate how we construct a T' given that the complex is exact. Recall that the canonical factorization of g expands to $Y \twoheadrightarrow \mathrm{Coker}(f) \twoheadrightarrow \mathrm{Im}(g) \hookrightarrow Z$. If $h : Y \to T$ is such that $hf = 0$ it factors through a morphism $T \to \mathrm{Coker}(f)$. Since $\mathrm{Coker}(f) \to \mathrm{Im}(g)$ is an isomorphism the composite $\mathrm{Coker}(f) \to Z$ is a monomorphism. In the diagram



the bottom row is a monomorphism since monomorphisms in an abelian category are universal (i.e. pushouts of monomorphisms are monomorphisms). We can take $T' = T \amalg^Y Z$; the rest of the argument is similarly dual to the earlier one.

Graded objects and complexes

Suppose $\mathcal{A}$ is an additive category and I is a set. An *I-graded object of $\mathcal{A}$* is an object X of $\mathcal{A}$ and a fixed isomorphism $X \simeq \bigoplus_{i \in I} X_i$. We will usually denote such objects by $X.$ or $(X_i)_{i \in I}$ if I needs to be made explicit. Each X_i is a subobject of X and is called the *degree i component* of X. A *morphism* $f : X. \to Y.$ of I-graded objects is a set of morphisms $f_i : X_i \to Y_i$ for all i; we will denote it by $f = f.$ Graded objects and their morphisms form a category which I will denote by $\mathrm{gr}(I, \mathcal{A})$. It can also be viewed as the category of functors $S \to \mathcal{A}$ where S is viewed as a discrete category (the only morphisms are identities). It is easily checked the $\mathrm{gr}(I, \mathcal{A})$ is additive: the zero object is $\bigoplus_i 0$ and the direct sum of $X.$ and $Y.$ is the graded object whose degree i component is $X_i \oplus Y_i$.

If $\mathcal{A}$ is preabelian so is $\mathrm{gr}(I, \mathcal{A})$: the kernel of $f. : X. \to Y.$ is the morphism $\mathrm{Ker}(f.)$ whose components are the canonical $\mathrm{Ker}(f_i) \to X_i$ and the cokernel has components $Y_i \to \mathrm{Coker}(f_i)$. Finally if $\mathcal{A}$ is abelian so is $\mathrm{gr}(I, \mathcal{A})$. I will leave all these verifications to you.

The case when I is a monoid is particularly useful. The most common cases are $I = \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}$ or $\mathbb{Z}^n$. If $i \in I$ and $X = \bigoplus_i X_i$ is an object of $\mathrm{gr}(I, \mathcal{A})$, the *shifted* object $X[i]$ is defined by $X[i]_j = X_{i+j}$. In other words $X[i]$ is the same object of $\mathcal{A}$, but the various degree components have been shifted by j. Shifting is a functor: if $f : X \to Y$ is a morphism in $\mathrm{gr}(I, \mathcal{A})$ then $f[i] : X[i] \to Y[i]$ the morphism whose degree j component is $f_{i+j} : X_{i+j} \to Y_{i+j}$. When $\mathcal{A}$ is abelian it is an *exact* functor: if

$$0 \to X \to Y \to Z \to 0$$

is an exact sequence in $\mathrm{gr}(I, \mathcal{A})$ then so is

$$0 \to X[i] \to Y[i] \to Z[i] \to 0$$

for any i.

We now specialize to the case $I = \mathbb{Z}$ and take $\mathcal{A}$ to be an abelian category. A *chain complex* of objects of $\mathcal{A}$ is an pair $(X., d)$ where $X.$ is a $\mathbb{Z}$-graded object of $\mathcal{A}$ and $d : X \to X[-1]$ is a morphism in $\mathrm{gr}(\mathbb{Z}, \mathcal{A})$ such that

$$X. \xrightarrow{d} X.[-1] \xrightarrow{d[-1]} X.[-2]$$

is zero. In other words the morphisms $d_{i-1}d_i : X_i \to X_{i-2}$ are zero for all $i \in \mathbb{Z}$. A *morphism* $f : (X., d^X_.) \to (Y., d^Y_.)$ is a morphism of $\mathbb{Z}$-graded complexes $f : X. \to Y.)$ such that

$$\begin{array}{ccc} X. & \xrightarrow{f} & Y. \\ {\scriptstyle d^X}\downarrow & & \downarrow{\scriptstyle d^Y} \\ X.[-1] & \xrightarrow{f[-1]} & Y.[-1] \end{array}$$

commutes. We denote by $C.(\mathcal{A})$ the category of chain complexes of objects of $\mathcal{A}$.

Concretely a chain complex is a set of commutative diagrams

$$\begin{array}{ccc}
X_i & \xrightarrow{f_i} & Y_i \\
{\scriptstyle d_i^X}\downarrow & & \downarrow{\scriptstyle d_i^Y} \\
X_{i-1} & \xrightarrow{f_{-1}} & Y_{i-1}
\end{array}$$

for all $i \in \mathbb{Z}$.

The dual notion is that of a *cochain complex* which is a pair $(X^\cdot, d)$ where $X^\cdot$ is a $\mathbb{Z}$-graded object of $\mathcal{A}$ and $d : X^\cdot \to X^\cdot[1]$ is a morphism such that $d[1]d = 0$. Morphisms are defined similarly, and the category of cochain complexes in $\mathcal{A}$ will be denoted by $C^\cdot(\mathcal{A})$.

Usually the grading of a cochain complex is written as a superscript, so that the degree i part of d is $d^i : X^i \to X^{i+1}$. In any case we will usually write X for $X_.$ or $X^\cdot$ if the meaning is clear, i.e. if X is known to be a chain or cochain complex.

Obviously a chain complex in $\mathcal{A}$ is a cochain complex in $\mathcal{A}^{\mathrm{op}}$. Alternatively, a chain complex $X_\cdot$ in $\mathcal{A}$ can be made into a cochain complex in $\mathcal{A}$ by setting $X^i = X_{-i}$, and *vice versa*. Thus everything we prove for chain complexes holds for cochain complexes. In what follows I will mostly use cochain complexes.

If (X, d) is a cochain complex the *cocycles* and *coboundaries* are the $\mathbb{Z}$-graded modules $Z = \mathrm{Ker}(d : X \to X[1])$ and $B = \mathrm{Im}(d[-1] : X[-1] \to X)$. Explictly,

$$Z^i = \mathrm{Ker}(d^i : X^i \to X^{i+1}), \qquad B^i = \mathrm{Im}(d^{i-1} : X^{i-1} \to X^i).$$

Of course $Z^\cdot$ and $B^\cdot$ are objects of $C^\cdot$, i.e. complexes in $\mathcal{A}$, but since the differential for both is 0 there is no point in regarding them as anything other than graded modules.

Since $dd[-1] = 0$, i.e. since

$$X[-1] \xrightarrow{d[-1]} X \xrightarrow{d} X[1] \tag{1}$$

is a complex in $C^{\cdot}(\mathcal{A})$ there is a canonical monomorphism $B^{\cdot} \to Z^{\cdot}$ of graded modules.

The (co)homology of the complex (1), i.e. the cokernel of $\mathrm{Im}(d[-1]) \to \mathrm{Ker}(d)$ is thus the cokernel of $B^{\cdot} \to Z^{\cdot}$, so that

$$0 \to B^{\cdot} \to Z^{\cdot} \to H^{\cdot}(X) \to 0$$

is exact. Heuristically we could write this as

$$H^{\cdot}(X) \simeq Z^{\cdot}/B^{\cdot}$$

as if everything were modules.

Let's now apply the proposition we proved earlier: the cokernel of $\mathrm{Im}(g) \to \mathrm{Ker}(g)$ is isomorphic to kernel of $\mathrm{Coker}(d[-1]) \to \mathrm{Im}(d)$, which we could write as an exact sequence

$$0 \to H^{\cdot}(X) \to X^{\cdot}/B^{\cdot} \to Z^{\cdot}[1] \to 0$$

where $X^{\cdot}/B^{\cdot} \to Z^{\cdot}$ is induced by d, which factors $X^{\cdot} \to X^{\cdot}/B^{\cdot} \to Z^{\cdot} \to X^{\cdot}[1]$.

We now apply all this mechanism to an exact sequence of complexes

$$0 \to A^{\cdot} \to B^{\cdot} \to C^{\cdot} \to 0.$$

Since the differential of a complex is in fact a morphism of complexes, we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^{\cdot} & \longrightarrow & B^{\cdot} & \longrightarrow & C^{\cdot} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_A} & & \downarrow{\scriptstyle d_B} & & \downarrow{\scriptstyle d_C} & & \\ 0 & \longrightarrow & A^{\cdot}[1] & \longrightarrow & B^{\cdot}[1] & \longrightarrow & C^{\cdot}[1] & \longrightarrow & 0. \end{array}$$

Let's write $Z^{\cdot}(A)$ and $B^{\cdot}(A)$ for the cocycles and coboundaries of A, and similarly for B and C. The snake lemma then gives us a 6-term exact sequence

$$\begin{aligned} 0 \to Z^{\cdot}(A) \to Z^{\cdot}(B) \to Z^{\cdot}(C) \to \\ \to A^{\cdot}[1]/B^{\cdot}(A[1]) \to B^{\cdot}[1]/B^{\cdot}(B[1]) \to C^{\cdot}[1]/B^{\cdot}(C[1]) \to 0 \end{aligned}$$

This can now be reassembled into a commutative diagram

$$\begin{array}{ccccccc} & & A^{\cdot}/B^{\cdot}(A) & \longrightarrow & B^{\cdot}/B^{\cdot}(B) & \longrightarrow & C^{\cdot}/B^{\cdot}(C) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z^{\cdot}(A)[1] & \longrightarrow & Z^{\cdot}(B)[1] & \longrightarrow & Z^{\cdot}(C)[1] & & \end{array}$$

in which the vertical morphisms are induced by the differentials of the complexes.

For any complex $X^\cdot$ the kernel of $X^\cdot/B^\cdot(X) \to Z^\cdot(X)[1]$ is isomorphic to the kernel of $X^\cdot/B^\cdot(X) \to X^\cdot[1]$, which by our earlier results is isomorphic to $H^\cdot(X)$. Likewise the cokernel of $X^\cdot/B^\cdot(X) \to Z^\cdot(X)[1]$ is isomorphic to the cokernel of $X^\cdot \to Z^\cdot(X)[1]$, which is $H^\cdot(X)[1]$. Therefore applying the snake lemma yet again to the last diagram yields a six-term exact sequence

$$H^\cdot(A) \to H^\cdot(B) \to H^\cdot(C) \to H^\cdot(A)[1] \to H^\cdot(B)[1] \to H^\cdot(C)[1].$$

In terms of components this is an infinite exact sequence

$$\to H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to H^{i+1}(B) \to H^{i+1}(C) \to$$

called the *long exact sequence* associated to the original short exact sequence of complexes. The morphism $H^i(C) \to H^{i+1}(A)$ is called the *connecting* or *Bockstein* morphism.

The formula for the connecting morphism comes from the same formula for the connecting morphism in the snake lemma. For simplicity we set

$$T = (B^\cdot/B^\cdot(B)) \times_{(C^\cdot/B^\cdot(C))} H^\cdot(C)$$

and consider

$$\begin{array}{ccccccccc}
 & & & & T & \longrightarrow & H^\cdot(C) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & A^\cdot/B^\cdot(A) & \longrightarrow & B^\cdot/B^\cdot(B) & \longrightarrow & C^\cdot/B^\cdot(C) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z^\cdot(A)[1] & \longrightarrow & Z^\cdot(B)[1] & \longrightarrow & Z^\cdot(C)[1] & & \\
 & & \downarrow & & & & & & \\
 & & H^\cdot(A)[1] & & & & & &
\end{array}$$

We recall that in this diagram there is a unique morphism $f : T \to Z^{\cdot}(A)[1]$ such that the composite $T \xrightarrow{f} Z^{\cdot}(A)[1] \to Z^{\cdot}(B)[1]$ is the same as $T \xrightarrow{p_1} B^{\cdot}/B^{\cdot}(B)$. Then $\partial : H^{\cdot}(C) \to H^{\cdot}(A)[1]$ is the unique morphism such that

$$\begin{array}{ccc} T & \xrightarrow{p_1} & H^{\cdot}(C) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \partial} \\ Z^{\cdot}(A)[1] & \longrightarrow & H^{\cdot}(A)[1] \end{array}$$

commutes.

In module categories ∂ is computed as follows. Suppose $x \in H^i(C)$ and pick an element $y \in B^{\cdot}/B^{\cdot}(B)$ such that y and x map to the same element of $C^{\cdot}/B^{\cdot}(C)$ under the respective maps (note, in particular that $(x, y) \in T$). The image of $d_B(y) \in Z^{i+1}(B)$ dies in $Z^{i+1}(C)$, so it comes from a unique element of $Z^{i+1}(A)$ whose image in $H^{i+1}(A)$ is $\partial(x)$.

From this we can deduce that the connecting morphism is functorial in the following sense: suppose

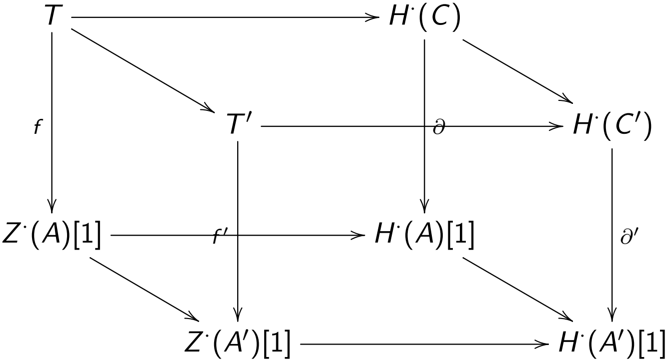
$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^\cdot & \longrightarrow & B^\cdot & \longrightarrow & C^\cdot & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A'^\cdot & \longrightarrow & B'^\cdot & \longrightarrow & C'^\cdot & \longrightarrow & 0 \end{array}$$

is a commutative diagram of cochain complexes in which the rows are exact. Then the square

$$\begin{array}{ccc} H^\cdot(C) & \xrightarrow{\partial} & H^\cdot(A)[1] \\ \downarrow & & \downarrow \\ H^\cdot(C') & \xrightarrow[\partial']{} & H^\cdot(A')[1] \end{array}$$

is commutative.

One way to prove this is to use the Freyd-Mitchell theorem to reduce to the case of module categories and then use the description given in that case. Another is to consider the cubical diagram



in which all faces are commutative except possibly for the right-hand side face, which is the on expressing the functoriality of the connecting morphism.

To show that the two compositions in the right-hand side face are equal it suffices to show that their pre-compositions with the epimorphism $T \to H^{\cdot}(C)$ are equal. This is easy to do; easier at any rate than typesetting the cubical diagram.