Homological Algebra
Lecture 6

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Additive functors

If $\mathcal{A}$ and $\mathcal{B}$ are additive functors, a functor $F : \mathcal{A} \to \mathcal{B}$ is *additive* if the map $\mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{B}}(F(X), F(Y))$ is a homomorphism for all X, Y in $\mathcal{A}$. If F is additive then $F(0) = 0$.

Lemma

A functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is additive if and only if it commutes with direct sums, i.e. the morphism

$$[F(i_1), F(i_2)] : F(X) \oplus F(Y) \to F(X \oplus Y)$$

is an isomorphism

Proof: If F is additive then applying F to the equality $1_{X \oplus Y} = i_1 p_1 + i_2 p_2$ yields

$$1_{F(X \oplus Y)} = F(1_{X \oplus Y}) = F(i_1 p_1 + i_2 p_2) = F(i_1)F(p_1) + F(i_2)F(p_2)$$

by additivity, and this shows that

$$(F(p_1), F(p_2)) : F(X \oplus Y) \to F(X) \oplus F(Y)$$

is inverse to $[F(i_1), F(i_2)]$.

Suppose conversely that F commutes with direct sums. We first recall that for any X in $\mathcal{A}$ the diagonal morphism

$$\Delta_X : X \to X \oplus X$$

i.e. the unique morphism such that $p_1 \Delta_X = p_2 \Delta_X = 1_X$ is given by

$$\Delta_X = i_1 + i_2.$$

Dually the codiagonal $\nabla_X : X \oplus X \to X$, which is the unique morphism such that $\nabla_X i_1 = \nabla_X i_2 = 1_X$ is given by

$$\nabla_X = p_1 + p_2$$

(I leave this as an exercise).

Finally we showed that if f and g are morphisms $X \to Y$, the direct sum $f \oplus g : X \oplus X \to Y \oplus Y$ is

$$f \oplus g = i_1 f p_1 + i_2 g p_2.$$

This formula, together with

$$p_1 i_1 = p_2 i_2 = 1, \qquad p_2 i_1 = p_1 i_2 = 0$$

show that

$$f + g = \nabla_Y (f \oplus g) \Delta_X.$$

Suppose now $F : \mathcal{A} \to \mathcal{B}$ commutes with direct sums, i.e. $F(X \oplus Y) \simeq F(X) \oplus F(Y)$ via the natural morphism. The universal properties of products and coproducts show that

$$F(\Delta_X) = \Delta_{F(X)}, \qquad F(\nabla_X) = \nabla_{F(X)}$$

for any X in $\mathcal{A}$. Furthermore

$$F(f \oplus g) = F(f) \oplus F(g)$$

for any $f, g : X \to Y$. Therefore

$$\begin{aligned} F(f+g) &= F(\Delta_Y (f \oplus g) \nabla_X) \\ &= \Delta_{F(Y)}(F(f) \oplus F(g)) \nabla_{F(X)} = F(f) + F(g). \end{aligned}$$

∎

In general, a functor $F : \mathcal{A} \to \mathcal{B}$ is *left exact* (resp. *right exact*) if it commutes with finite limits (resp. finite colimits). Since any finite limit is the equalizer of two morphisms between two products, F is left exact if and only if it commutes with finite products and equalizers. Dually, F is right exact if and only if it commutes with finite coproducts and coequalizers.

In an abelian category finite products and finite coproducts coincide, so we conclude that a left or right exact functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is additive.

Lemma

Suppose $\mathcal{A}$ and $\mathcal{B}$ are abelian. A functor $F : \mathcal{A} \to \mathcal{B}$ is left exact if and only if for every exact sequence

$$0 \to A \to B \to C$$

the sequence

$$0 \to F(A) \to F(B) \to F(C)$$

is also exact.

Proof: Suppose F is left exact and $f : B \to C$ is the morphism in the exact sequence. Then the exact sequence identifies $A \to B$ with the equalizer of f and the zero morphism $B \to C$, so by hypothesis $F(A)$ is the equalizer of $F(f)$ and $0 = F(0)$; this says that the second sequence is exact.

Suppose conversely that exactness of the first sequence implies exactness of the second. If f and $g : B \to C$ are two morphisms the equalizer of f and g is the kernel of $f - g$, so the hypothesis implies that F commutes with equalizers.

To finish we must show that F commutes with finite products, and by induction it suffices to treat the case of products of two objects. Given A and B in $\mathcal{A}$, consider the exact sequence

$$0 \to A \xrightarrow{i_1} A \oplus B \xrightarrow{p_2} B \to 0.$$

By hypothesis the lower row in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F(A) & \longrightarrow & F(A) \oplus F(B) & \longrightarrow & F(B) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & F(A) & \longrightarrow & F(A \oplus B) & \longrightarrow & F(B) & &
\end{array}$$

is exact. The snake lemma then shows that $F(A) \oplus F(B) \to F(A \oplus B)$ is an isomorphism. ■

Many other important functors are additive. For example if $\mathcal{A}$ is an additive category, bilinearity of composition shows that the functors

$$T \mapsto \mathrm{Hom}_{\mathcal{A}}(X, T), \qquad T \mapsto \mathrm{Hom}_{\mathcal{A}}(T, X)$$

are additive. If $\mathcal{A}$ is abelian the translation functor

$$C^{\cdot}(\mathcal{A}) \to C^{\cdot}(\mathcal{A}) \qquad X^{\cdot} \mapsto X^{\cdot}[1]$$

is additive (evidently – the shift just re-indexes components of a morphism). It will be important the the functors

$$Z^{\cdot},\ B^{\cdot},\ H^{\cdot} : C^{\cdot}(\mathcal{A}) \to \mathrm{gr}(\mathbb{Z}, \mathcal{A})$$

are all additive. It suffices to show that they commute with finite direct sums (i.e. finite products, or coproducts). The case of $Z^{\cdot}$ is simple because it is a limit (in fact an equalizer) and thus commutes with direct sums (interpreted as products). Since $B^{\cdot}$ is a cokernel (i.e. a colimit, in fact a coequalizer) it commutes with direct sums interpreted as coproducts. Finally $H^{\cdot}$ can be interpreted as either a cokernel or a kernel, so it too commutes with direct sums.

Resolutions

We now fix an abelian category $\mathcal{A}$; unless otherwise stated objects and morphisms are in $\mathcal{A}$.

A *chain resolution* of an object M is an exact complex

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0 \to 0 \to \cdots \tag{1}$$

where P_0 is in degree 0. A *cochain resolution* of M is an exact complex

$$\cdots \to 0 \to 0 \to M \to I^0 \to I^1 \to I^2 \to \cdots \tag{2}$$

where again I^0 is in degree zero. Evidently a chain resolution of M in $\mathcal{A}$ is a cochain resolution in $\mathcal{A}^{\mathrm{op}}$ and *vice versa*. We will frequently write $P_\cdot \to M$ for a chain resolution of M, and $M \to I^\cdot$ for a cochain resolution.

A chain resolution is a a *projective resolution* if the P_i are all projective. Likewise a cochain resolution is an *injective resolution* if the I_i are injective.

Homotopies

Chain and cochain resolutions are evidently not unique. To compare them we will make use of the following notion. Suppose f and $g : X^\cdot \to Y^\cdot$ are morphisms of cochain complexes. A *homotopy* between f and g is a morphism of graded objects $h : X^\cdot \to Y^\cdot[-1]$ (*not* of complexes) such that

$$f - g = d_Y[-1]h + h[1]d_X. \tag{3}$$

Morphisms $f, g : X^\cdot \to Y^\cdot$ are *homotopic* if there exists a homotopy between f and g. A *null homotopy* of a morphism $f : X^\cdot \to Y^\cdot$ is a homotopy between f and the 0 morphism, and f is *null homotopic* if it has a null homotopy.

Proposition

If $f, g : X^\cdot \to Y^\cdot$ are homotopic, $H^\cdot(f) = H^\cdot(g)$.

Proof: Since $H^{\cdot}$ is an additive functor it suffices to show that if f is null homotopic then $H^{\cdot}(f) = 0$. The assumption says that there is a morphism $h : X^{\cdot} \to Y^{\cdot}[-1]$ such that $f = d_Y[-1]h + h[1]d_X$. Observe that $d_Y[-1]h$ and $h[1]d_X$ are both morphisms $X^{\cdot} \to Y^{\cdot}$ as $\mathbb{Z}$-graded objects. I will show that they induce the zero map $H^{\cdot}(X) \to H^{\cdot}(Y)$; since $H^{\cdot}$ is additive this will show that $H^{\cdot}(f) = 0$.

(1) $H^{\cdot}(h[1]d_X) = 0$: since $H^{\cdot}(X)$ is a quotient of $Z^{\cdot}(X)$ it suffices to show that $h[1]d_X$ annihilates $Z^{\cdot}$, but this is obvious since by definition $Z^{\cdot}(X)$ is the kernel of $d_X : X^{\cdot} \to X^{\cdot}[1]$.

(2) $H^{\cdot}(d_Y[-1]h) = 0$: since $H^{\cdot}(Y^{\cdot})$ is subobject of $Y^{\cdot}/B^{\cdot}(Y)$ it suffices to show that the morphism $Z^{\cdot}(X)[-1] \to Y^{\cdot}/B^{\cdot}(Y)$ induced by d_Y is zero, but this is obvious since by definition $B^{\cdot}(Y)$ is the image of $d_Y[-1] : Y^{\cdot}[-1] \to Y^{\cdot}$. ■

If f and $g : X^{\cdot} \to Y^{\cdot}$ are morphisms we will write $f \sim g$ to indicate that f and g are homotopic. This is easily seen to be an equivalence relation:

- If $f = g$ we can take $h = 0$, so $f \sim f$.
- If $f \sim g$ then $f - g = d_Y[-1]h + h[1]d_X$ and then $g - f = d_Y[-1](-h) + (-h)[1]d_X$, whence $g \sim f$.
- Suppose $f \sim g$ and $g \sim h$, and let ℓ and m be such that

$$f - g = d_Y[-1]\ell + \ell[1]d_X, \qquad g - h = d_Y[-1]m + m[1]d_X.$$

Then

$$f - h = d_Y[-1](\ell + m) + (\ell + m)[1]d_X.$$

and thus $f \sim h$.

Suppose finally that $f, f' : X^{\cdot} \to Y^{\cdot}$ and $g, g' : Y^{\cdot} \to Z^{\cdot}$ are morphisms. If $f \sim f'$ and $g \sim g'$ then $gf \sim g'f'$. Suppose $h : X^{\cdot} \to Y^{\cdot}[-1]$ and and $h' : Y^{\cdot} \to Z^{\cdot}[-1]$ are such that

$$f' - f = d_Y[-1]h + h[1]d_X, \qquad g' - g = d_Z[-1]h' + h'[1]d_Y.$$

Then

$$g'(f'-f) = g'(d_Y[-1]h + h[1]d_X) = d_Z[-1]g'h + g'h[1]d_X$$
$$(g'-g)f = (d_Z[-1]h' + h'[1]d_Y)f = d_Z[-1]h'f + h'f[1]d_X$$

since g' and f are morphisms of complexes. Adding, we find

$$g'f' - gf = d_Z(g'h + h'f) + (g'h + h'f)d_X$$

which shows that $g'f' \sim gf$.

Remark: there is some interesting structure here that we are ignoring. For any objects $X^\cdot$, $Y^\cdot$ of $C^\cdot(\mathcal{A})$ we can make $\mathrm{Hom}_{C^\cdot(\mathcal{A})}(X^\cdot, Y^\cdot)$ into a category by taking as morphisms $f \to g$ chain maps $h : X^\cdot \to Y^\cdot[-1]$ such that $g - f = d_Y h + hd_X$. Our earlier proof that homotopy is an equivalence relation then shows that the axioms of a category are satisfied and that every morphism in $\mathrm{Hom}_{C^\cdot(\mathcal{A})}(X^\cdot, Y^\cdot)$ is an isomorphism. This in turn means that $C^\cdot(\mathcal{A})$ itself has the structure of a 2-category (the Hom sets have a category structure compatible with composition).

Proposition

Suppose $\mathcal{A}$ is an abelian category, M and N are objects of $\mathcal{A}$, $M \to M^{\cdot}$ is a cochain resolution, $N \to I^{\cdot}$ is an injective resolution and $f : M \to N$ is a morphism. There is a morphism of complexes $\bar{f} : M^{\cdot} \to I^{\cdot}$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & M^0 & \longrightarrow & M^1 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}^0} & & \downarrow{\scriptstyle \bar{f}^1} & & \\ 0 & \longrightarrow & N & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots \end{array}$$

is commutative. If $\tilde{f} : M^{\cdot} \to I^{\cdot}$ is another such morphism, $\bar{f}$ and $\tilde{f}$ are homotopic.

In other words $\bar{f}$ is unique up to homotopy. There is a similar assertion for projective resolutions:

Proposition

Suppose $\mathcal{A}$ is an abelian category, M and N are objects of $\mathcal{A}$, $M. \to M$ is a chain resolution, $P. \to N$ is an projective resolution and $f : N \to M$ is a morphism. There is a morphism of complexes $\bar{f} : P. \to M.$ such that

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \bar{f}_2} & & \downarrow{\scriptstyle \bar{f}_1} & & \downarrow{\scriptstyle f} & & \\ \cdots & \longrightarrow & M_2 & \longrightarrow & M_1 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

is commutative. If $\tilde{f} : P. \to M.$ is another such morphism, $\bar{f}$ and $\tilde{f}$ are homotopic.

The two propositions are proven in the same way, the proof of the second being in essence the proof of the first with the arrows reversed. I will prove the existence of $\bar{f}$ and its uniqueness up to homotopy in the first case, the second begin dual to the first.

We construct the $\bar{f}^i$ by induction. Since $M \to M^0$ is a monomorphism I^0 is an injective module the composite $M \to N \to I^0$ factors through $M \to M^0$, via a homomorphism $\bar{f}^0 : M^0 \to I^0$. By construction the square

$$\begin{array}{ccc} M & \longrightarrow & M^0 \\ \Big\downarrow f & & \Big\downarrow \bar{f}^0 \\ N & \longrightarrow & I^0 \end{array}$$

is commutative.

If $\bar{f}^i$ has been constructed for $0 \leq i \leq n$ then consider the diagram with exact rows

$$\begin{array}{ccccc} 0 & \longrightarrow & M^n/\bar{M}^{n-1} & \longrightarrow & M^{n+1} \\ & & \Big\downarrow \bar{f}^n & & \\ 0 & \longrightarrow & I^n/\bar{I}^{n-1} & \longrightarrow & I^{n+1} \end{array}$$

where $\bar{M}^{n-1}$ and $\bar{I}^{n-1}$ are the images of M^{n-1} and I^{n-1} in M^n and I^n.

Since $M^n/\bar{M}^{n-1} \to M^{n+1}$ is an injective homomorphism and I^{n+1} is an injective module the composite $M^n/\bar{M}^{n-1} \to I^n/\bar{I}^{n-1} \to I^{n+1}$ factors through $M^n/\bar{M}^{n-1} \to M^{n+1}$ via some homomorphism $\bar{f}^{n+1} : M^{n+1} \to I^{n+1}$. Then

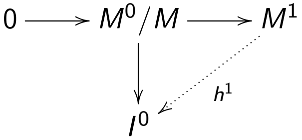
$$\begin{array}{ccc} M^n & \longrightarrow & M^{n+1} \\ \downarrow{\scriptstyle \bar{f}^n} & & \downarrow{\scriptstyle \bar{f}^{n+1}} \\ I^n & \longrightarrow & I^{n+1} \end{array}$$

is commutative. This completes the recursion.

We now show that if $\tilde{f} : M^\cdot \to I^\cdot$ is another morphism of complexes with the same property then $\bar{f}$ and $\tilde{f}$ are homotopic. Given such a $\tilde{f}$ there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & M^0 & \xrightarrow{d_M^0} & M^1 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \bar{f}^0 - \tilde{f}^0} & & \downarrow{\scriptstyle \bar{f}^1 - \tilde{f}^1} & & \\ 0 & \longrightarrow & N & \longrightarrow & I^0 & \xrightarrow{d_I^0} & I^1 & \longrightarrow & \cdots \end{array}$$

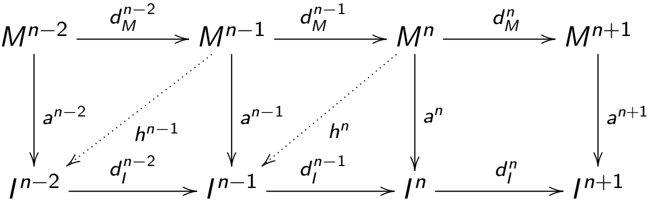
which shows that $\bar{f}^0 - \tilde{f}^0$ annihilates $M \subseteq M^0$. In what follows we set $a^i = \bar{f}^i - \tilde{f}^i$ and consider the diagram



of solid arrows. The dotted arrow exists since the top row is exact and I^0 is injective. Let $h^0 : M^0 \to 0$ be the zero map (the only possiblity!) Since $M^0/M \to M^1$ and $M^0/M \to I^0$ are induced by d_M^0 and $a^0 = \bar{f}^0 - \tilde{f}^0$ we have

$$\bar{f}^0 - \tilde{f}^0 = a^0 = d_I^0 h^0 + h^1 d_M^0.$$

Suppose that $h^i : M^i \to I^{i-1}$ has been constructed for $0 \leq i \leq n$. Then there is a commutative diagram of solid arrows



in which the vertical arrows are $a^i \bar{f}^i - \tilde{f}^i : M^i \to I^i$ and

$$a^{n-1} = h^n d_M^{n-1} + d_J^{n-2} h^{n-1}.$$

Then

$$
\begin{aligned}
(a^n - d_I^{n-1}h^n)d_M^{n-1} &= a^n d_M^{n-1} - d_I^{n-1} h^n d_M^{n-1} \\
&= a^n d_M^{n-1} - d_I^{n-1}(a^{n-1} - d_I^{n-2} h^{n-1}) \\
&= (a^n d_M^{n-1} - d_I^{n-1} a^{n-1}) - d_I^{n-1} d_I^{n-2} h^{n-1} \\
&= 0
\end{aligned}
$$

shows that $a^n - d_I^{n-1}h^n : M^n \to I^n$ factors through a homomorphism $j : M^n/\bar{M}^{n-1} \to I^n$, which as before $\bar{M}^{n-1}$ is the image of M^{n-1} in M^n. We now have a diagram

$$
\begin{array}{ccccc}
0 & \longrightarrow & M^n/\bar{M}^{n-1} & \longrightarrow & M^{n+1} \\
 & & \downarrow j & \swarrow h^{n+1} & \\
 & & I^n & &
\end{array}
$$

in which the top row is exact, and the dotted arrow can be filled in because I^n is injective.

Since j and $M^n/\bar{M}^{n-1} \to M^{n+1}$ are induced by $\bar{f}^n - \tilde{f}^n - d_I^{n-1}h^n$ and d_M^n,

$$\bar{f}^n - \tilde{f}^n - d_I^{n-1}h^n = h^{n+1}d_M^n$$

and therfore

$$\bar{f}^n - \tilde{f}^n = d_I^{n-1}h^n + h^{n+1}d_M^n$$

as desired. ∎