Homological Algebra
Lecture 7

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Derived Functors

Suppose $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $F : \mathcal{A} \to \mathcal{B}$ is a functor. Recall that F is left exact (resp. right exact) if it commutes with finite limits (resp. finite colimits). We showed that F is left exact if and only if for every exact sequence

$$0 \to M' \to M \to M''$$

in $\mathcal{A}$ the sequence.

$$0 \to F(M') \to F(M) \to F(M'')$$

is also exact. Dually, F is right exact if only if for every exact sequence

$$M' \to M \to M'' \to 0$$

in $\mathcal{A}$ the sequence

$$F(M') \to F(M) \to F(M'') \to 0$$

is also exact.

If $\mathcal{A}$ is abelian so is $\mathcal{A}^{\text{op}}$, and then the above exactness conditions for a functor $F : \mathcal{A}^{\text{op}} \to \mathcal{B}$ turn out to mean the following: F is left exact if and only if for every exact sequence

$$M' \to M \to M'' \to 0$$

in $\mathcal{A}$ the sequence

$$0 \to F(M'') \to F(M) \to F(M')$$

is exact; likewise F is right exact if and only if for every exact sequence

$$0 \to M' \to M \to M''$$

the sequence

$$F(M'') \to F(M) \to F(M') \to 0$$

is exact.

Particularly important cases are the following:

(1) Suppose $\mathcal{A}$ is an abelian category and N is a fixed object of $\mathcal{A}$; then the functor

$$F : \mathcal{A} \to \mathbf{Mod}_{\mathbb{Z}} \qquad F(M) = \mathrm{Hom}_{\mathcal{A}}(N, M)$$

is left exact.

(2) Suppose A a ring with identity and N is a fixed *right* A-module, the functor

$$F : \mathbf{Mod}_A \to \mathbf{Mod}_{\mathbb{Z}} \qquad F(M) = N \otimes_A M$$

is right exact.

In general if

$$0 \to A \to B \to C \to 0$$

is exact and F is left exact, the sequence

$$0 \to F(A) \to F(B) \to F(C) \to 0$$

is *not* exact.

The aim of the theory of derived functors is something like the following: if $F : \mathcal{A} \to \mathcal{B}$ is left exact, to construct a sequence of functors $R^n F : \mathcal{A} \to \mathcal{B}$ for all $n \geq 0$ and a functorial isomorphism $R^0 F \simeq F$ so that for any short exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$ there is a long exact sequence

$$\cdots \to R^n F(A) \to R^n F(B) \to R^n F(C) \to R^{n+1} F(A) \to \cdots$$

for all $n \geq 0$. The $R^n F$ are called the *right derived functors* of F. If on the other hand $F : \mathcal{A} \to \mathcal{B}$ is right exact I will construct functors $L_n F : \mathcal{A} \to \mathcal{B}$ and a functorial isomorphism $L_0 F \simeq F$ as before, and a long exact sequence

$$\cdots \to L_n F(A) \to L_n F(B) \to L_n F(C) \to L_{n-1} F(A) \to \cdots$$

The $L_n F$ are called the *left derived functors* of F.

Finally if $F : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$ is left (resp. right) exact we construct right (resp. left) derived functors as before, but now the "direction" of the long exact sequence is reversed. The constructions in the four different cases are entirely similar and I will work out in detail the case of a left exact $F : \mathcal{A} \to \mathcal{B}$. When F is left (resp. right) exact it is essential that $\mathcal{A}$ has enough injectives (resp. enough projectives). Now you know what the "enough" means. The construction of the R^nF will use the whole machinery of complexes, homology and homotopy we have so far developed.

Let's begin with some general observations about complexes and functors. Suppose $F : \mathcal{A} \to \mathcal{B}$ is an additive functor between abelian categories. If $(X^{\cdot}, d)$ is an object of $C^{\cdot}(\mathcal{A})$, the category of cochain complexes in $\mathcal{A}$ then $(F(X^{\cdot}), F(d))$ is an object of $C^{\cdot}(\mathcal{B})$: in fact $F(X^{\cdot})$ is clearly a $\mathbb{Z}$-graded object of $\mathcal{A}$, $d : X^{\cdot} \to X^{\cdot}[1]$ induces $F(d) : F(X^{\cdot}) \to F(X^{\cdot})[1]$ and $d[1]d = 0$ implies $F(d[1])F(d) = 0$.

Suppose next that $X^\cdot$ and $Y^\cdot$ are cochain complexes in $\mathcal{A}$, f and $g : X^\cdot \to Y^\cdot$ are morphisms of complexes and finally $h : X^\cdot \to Y^\cdot[-1]$ is a homotopy from f to g, i.e.

$$f - g = d_Y[-1]h + h[1]d_X.$$

If $F : \mathcal{A} \to \mathcal{B}$ is an *additive functor*, $F(h)$ is a homotopy from $F(f)$ to $F(g)$. In fact applying F to the previous equation and appealing to the additivity of F we find that

$$\begin{aligned} F(f) - F(g) &= F(f - g) \\ &= F(d_Y[-1]h + h[1]d_X) \\ &= F(d_Y)[-1]F(h) + F(h)[1]F(d_X). \end{aligned}$$

In particular this holds if F is left or right exact.

Assume now that $\mathcal{A}$ has enough injectives and that $F : \mathcal{A} \to \mathcal{B}$ is left exact. Suppose M is an object of $\mathcal{A}$ and $M \to I^{\cdot}$ is an injective resolution. By our earlier remarks $F(I^{\cdot})$ is a complex in $\mathcal{B}$. I will show that the objects $H^n(F(I^{\cdot}))$ are independent of the choice of $M \to I^{\cdot}$ up to canonical isomorphism. We can then define $R^nF(M) = H^n(F(I^{\cdot}))$ for any injective resolution $M \to I^{\cdot}$. The left exactness of F will imply that $R^0F \simeq F$, and finally I will show that $M \mapsto R^nF(M)$ is a functor $\mathcal{A} \to \mathcal{B}$.

Suppose $M \to J^{\cdot}$ is another injective resolution. By an earlier proposition there is a morphism $f : I^{\cdot} \to J^{\cdot}$ of complexes, unique up to homotopy. Applying F yields a morphism $F(f) : F(I^{\cdot}) \to F(J^{\cdot})$ in $C^{\cdot}(\mathcal{B})$. If $f' : I^{\cdot} \to J^{\cdot}$ is another morphism, $f \sim f'$ and thus $F(f) \sim F(f')$ since F is additive, being in fact left exact. From this we conclude that $F(f)$ and $F(f')$ define the *same* morphism $H^n(F(I^{\cdot})) \to H^n(F(J^{\cdot}))$. Thus to any two choices of injective resolution $M \to I^{\cdot}$, $M \to J^{\cdot}$ there is a canonical morphism $H^n(F(I^{\cdot})) \to H^n(F(J^{\cdot}))$.

In particular since $M \to I^\cdot$ is an injective resolution there is a morphism $g : J^\cdot \to I^\cdot$ unique up to homotopy. As before applying F yields a morphism $F(g) : F(J^\cdot) \to F(I^\cdot)$ and the morphisms $H^n(F(J^\cdot)) \to H^n(F(I^\cdot))$ are independent of the choice of g.

I claim that $H^n(F(f))$ and $H^n(F(g))$ are inverse isomorphisms. It suffices to show that $F(g) \circ F(f) \sim 1_{F(I^\cdot)}$ and $F(f) \circ F(g) \sim 1_{F(J^\cdot)}$, and this follows by application of F if we can show that

$$g \circ f \sim 1_{I^\cdot} \quad \text{and} \quad f \circ g \sim 1_{J^\cdot}.$$

However $g \circ f$ and $1_{I^\cdot}$ are both morphisms $I^\cdot \to I^\cdot$ and $M \to I^\cdot$ is an injective resolution, so they are homotopic by an earlier proposition. Similarly $f \circ g$ and $1_{J^\cdot}$ are both morphisms $J^\cdot \to J^\cdot$, so they are homotopic. This proves the claim.

We have shown that for all $n \geq 0$ the objects $H^n(F(I^\cdot))$ of $\mathcal{B}$ are independent of the choice of injective resolution $M \to I^\cdot$ up to canonical isomorphism, so we define can define $R^nF(M) = H^n(F(I^\cdot))$. We next construct an isomorphism $R^0F(M) \simeq F(M)$, By construction $R^0F(M)$ is the H^0 of the complex

$$\cdots \to 0 \to 0 \to F(I^0) \to F(I^1) \to \cdots$$

with $F(I^0)$ in degree zero, so

$$R^0F(M) \simeq \mathrm{Ker}(F(I^0) \to F(I^1)).$$

However F is left exact, so

$$0 \to F(M) \to F(I^0) \to F(I^1)$$

is exact as well. This yields the isomorphism $R^0F(M) \simeq F(M)$.

It remains to show that $M \mapsto R^nF(M)$ defines a functor $\mathcal{A} \to \mathcal{B}$ and that the isomorphism $R^0F(M) \xrightarrow{\sim} F(M)$ is functorial. Suppose $f : M \to N$ is a morphism in $\mathcal{A}$ and choose injective resolutions $M \to I^\cdot$, $N \to J^\cdot$. Since the latter is an injective resolution an earlier proposition yields a morphism $\bar{f} : I^\cdot \to J^\cdot$ of complexes, unique up to homotopy. Applying F yields a morphism $F(\bar{f}) : F(I^\cdot) \to F(J^\cdot)$, whence morphisms

$$R^nF(M) = H^n(F(I^\cdot)) \to H^n(F(J^\cdot)) = R^nF(N)$$

in $\mathcal{B}$ which are independent of the choice of $\bar{f}$. We take these homomorphisms as $R^nF(f) : R^nF(M) \to R^n(F(N))$. We must show that $R^nF(1_M) = 1_{R^n(M)}$ and that $R^nF(g \circ f) = R^nF(g) \circ R^nF(f)$ for composable morphisms f and g. I will carry out the latter since the former is elementary.

Suppose $f : M \to N$ and $g : N \to P$ are morphisms in $\mathcal{A}$ and $M \to I^\cdot$, $N \to J^\cdot$ and $P \to K^\cdot$ are all injective resolutions. By the proposition we get morphisms $\bar{f} : I^\cdot \to J^\cdot$, $\bar{g} : J^\cdot \to K^\cdot$ and applying it it $h = g \circ f : M \to P$ yields $\bar{h} : I^\cdot \to K^\cdot$. By definition $R^nF(f)$, $R^nF(g)$ and $R^nF(g \circ f)$ are the results of applying H^n to the morphisms $F(\bar{f}) : F(I^\cdot) \to F(J^\cdot)$, $F(\bar{g}) : F(J^\cdot) \to F(K^\cdot)$ and $F(\bar{h}) : F(I^\cdot) \to F(K^\cdot)$. However $\bar{g} \circ \bar{f}$ and $\bar{h}$ are both morphisms $I^\cdot \to K^\cdot$ and are homotopic, so $F(\bar{g}) \circ F(\bar{f})$ and $F(\bar{h})$ are also homotopic, which shows that $R^nF(g \circ f) = R^nF(g) \circ R^nF(f)$.

Finally, with the above notation the morphism $R^0F(M) \to R^0F(N)$ is $H^0(F(\bar{f}))$. Since the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & F(M) & \longrightarrow & F(I^0) & \longrightarrow & F(I^1) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F(N) & \longrightarrow & F(J^0) & \longrightarrow & F(J^1) \end{array}$$

is commutative the square

$$\begin{array}{ccc} F(M) & \longrightarrow & R^0F(M) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle R^0F(f)} \\ F(N) & \longrightarrow & R^0F(N) \end{array}$$

is commutative, which was to be shown.

The construction and arguments are the same for right exact functors $\mathcal{A} \to \mathcal{B}$, except that instead of instead of injective resolutions we use projective ones. In this case of course we have to assume that $\mathcal{A}$ has enough projectives, and the construction is as follows: if $P. \to M$ is a projective resolution the objects $H_n(F(P.))$ are independent of the choice of $P. \to M$ up to canonical isomorphism and we define

$$L_nF(M) = H_n(F(P.)).$$

This defines a functor $\mathcal{A} \to \mathcal{B}$. Since F is right exact the exactness of

$$P_1 \to P_0 \to M \to 0$$

implies that

$$F(P_1) \to F(P_0) \to F(M) \to 0$$

is exact, and as before this shows that $F(M) \simeq L_0F(M)$. Finally the latter isomorphism is functorial in M.

Finally the constructions are the same for left or right exact functors $F : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$. When F is left exact we must assume that $\mathcal{A}$ has enough projectives and the right derived functors of F are defined by $R^nF(M) = H^n(F(P.))$ for any projective resolution $P. \to M$. Similarly if F is right exact we must assume that $\mathcal{A}$ has enough injectives and then the left derived functors of F are $L_nF(M) = H_n(F(I^{\cdot}))$ for any injective resolution. I leave the details to you.

The remaining bit of the picture is the long exact sequence of derived functors. In the case where $F : \mathcal{A} \to \mathcal{B}$ is left exact and $\mathcal{A}$ has enough injectives, the assertion is that if

$$0 \to A \to B \to C \to 0$$

is an exact sequence in $\mathcal{A}$, there is a long exact sequence

$$\cdots \to R^n F(A) \to R^n F(B) \to R^n F(C) \xrightarrow{\partial} R^{n+1} F(A) \to \cdots$$

in $\mathcal{B}$. The main point is the following lemma:

Lemma

For any short exact sequence

$$0 \to A \to B \to C \to 0$$

of left A-modules and injective resolutions $A \to I'^{\cdot}$, $C \to I''^{\cdot}$ there is an injective resolution $B \to I^{\cdot}$ and an exact sequence of complexes

$$0 \to I'^{\cdot} \to I^{\cdot} \to I''^{\cdot} \to 0$$

such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I'^{0} & \longrightarrow & I^{0} & \longrightarrow & I''^{0} & \longrightarrow & 0 \end{array}$$

commutes.

Proof (sketch): We set $I^0 = I'^0 \oplus I''^0$. Since I'^0 is an injective A-module and $A \to B$ is an injective homomorphism, the composite $A \to I'^0 \to I^0$ factors through a homomorphism $B \to I^0$. There is also a homomorphism $B \to C \to I''^0 \to I^0$, and we take $B \to I^0$ to be the sum of these two homomorphisms. This yields the last diagram in the lemma. It is easily checked that the bottom row is exact, and that the middle vertical arrow is injective follows from the snake lemma. We can identify B with its image in I^0 and similarly for A and C. Repetition of the argument then leads to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I'^0/A & \longrightarrow & I^0/B & \longrightarrow & I''^0/C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I'^1 & \longrightarrow & I^1 & \longrightarrow & I''^1 & \longrightarrow & 0 \end{array}$$

with exact rows and the vertical arrows are injective. Repeat ad infinitum.■

Since each I'^n is injective the exact sequence

$$0 \to I'^{\cdot} \to I^{\cdot} \to I''^{\cdot} \to 0$$

in the lemma is term-by-term split (in fact this follows from the construction). Since F is additive it commutes with direct sums, and it follows that

$$0 \to F(I'^{\cdot}) \to F(I^{\cdot}) \to F(I''^{\cdot}) \to 0$$

is also an exact sequence of complexes. The associated long exact sequence of cohomology is by definition the long exact sequence

$$\cdots \to R^nF(A) \to R^nF(B) \to R^nF(C) \xrightarrow{\partial} R^{n+1}F(A)$$

of right derived functors of F. As before the morphism ∂ is called the *connecting morphism* or the *Bockstein morphism*.

We explain, finally that the long exact sequence is "functorial" in the following sense: suppose $\mathcal{A}$ and $F : \mathcal{A} \to \mathcal{B}$ are as above and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

is a commutative diagram with exact rows. Then

$$\begin{array}{ccccccccc} \longrightarrow & R^nF(A) & \longrightarrow & R^nF(B) & \longrightarrow & R^nF(C) & \xrightarrow{\partial} & R^{n+1}F(A) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & R^nF(A') & \longrightarrow & R^nF(B') & \longrightarrow & R^nF(C') & \xrightarrow{\partial} & R^{n+1}F(A') & \longrightarrow \end{array}$$

is commutative, where all morphism except the Bocksteins are induced by functoriality. In fact the left and middle squares are commutative since each R^nF is a functor, so the only question is that of the right one.

One way of seeing this is as follows. Choose injective resolutions $A \to I^{\cdot}$, $C \to K^{\cdot}$, $A' \to I^{'\cdot}$, $C \to K^{'\cdot}$ and let and let $B \to J^{\cdot}$, $B' \to J^{'\cdot}$ be the injective resolutions provided by the last lemma. We then have exact sequences of complexes

$$0 \to I^{\cdot} \to J^{\cdot} \to K^{\cdot} \to 0$$
$$0 \to I^{'\cdot} \to J^{'\cdot} \to K^{'\cdot} \to 0$$

that are term-by-term split. By an earlier lemma the morphisms $A \to A'$, $B \to B'$ and $C \to C'$ give rise to morphisms $I^{\cdot} \to I^{'\cdot}$, $J^{\cdot} \to J^{'\cdot}$ and $K^{\cdot} \to K^{'\cdot}$, so we now have a diagram of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^{\cdot} & \longrightarrow & J^{\cdot} & \longrightarrow & K^{\cdot} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^{'\cdot} & \longrightarrow & J^{'\cdot} & \longrightarrow & K^{'\cdot} & \longrightarrow & 0 \end{array}$$

with exact rows. The problem is that we don't know that the squares commute, and in fact this will usually not be the case.

We can however make the following observation. Both $A \to I^{\cdot}$ and $B' \to J^{'\cdot}$ are injective resolutions, and the composed morphisms

$$A \to A' \to B', \qquad A \to B \to B'$$

are equal. It follows that the composed morphisms of complexes

$$I^{\cdot} \to I^{'\cdot} \to J^{'\cdot}, \qquad I^{\cdot} \to J^{\cdot} \to J^{'\cdot}$$

are homotopic. In other words that diagram on the last slide is commutative up to homotopy. We then retrace all the arguments in the last two lectures concerning the functoriality of the Bockstein to conclude that this weaker condition suffices to prove that

$$\begin{array}{ccccccccc}
\longrightarrow & R^nF(A) & \longrightarrow & R^nF(B) & \longrightarrow & R^nF(C) & \xrightarrow{\partial} & R^{n+1}F(A) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & R^nF(A') & \longrightarrow & R^nF(B') & \longrightarrow & R^nF(C') & \xrightarrow{\partial} & R^{n+1}F(A) & \longrightarrow
\end{array}$$

is commutative.

There are easier ways of making this argument that we will get to later. For now we record two properties of derived functors, which we state for left exact $F : \mathcal{A} \to \mathcal{B}$.

The first is the the $R^nF : \mathcal{A} \to \mathcal{B}$ are all additive. This follows from the definition: if $M \to I^\cdot$ and $N \to J^\cdot$ are injective resolutions then $M \oplus N \to I^\cdot \oplus J^\cdot$ is evidently an injective resolution, and then

$$\begin{aligned} R^nF(M \oplus N) &= H^n(F(I^\cdot \oplus J^\cdot)) \simeq H^n(F(I^\cdot) \oplus F(J^\cdot)) \\ &\simeq H^n(F(I^\cdot)) \oplus H^n(F(J^\cdot)) = R^nF(M) \oplus R^nF(N) \end{aligned}$$

since F and H^n are additive.

The second is that if I is injective then $R^nF(I) = 0$ for all $n > 0$. This is clear since

$$\cdots \to 0 \to I \to I \to 0 \to 0 \to \cdots$$

is an injective resolution of I.

Similarly if F is right exact and P is projective then $L_nF(P) = 0$ for all $n > 0$.