Homological Algebra
Lecture 8

Richard Crew

Summer 2021

∂-functors

We have seen how universal properties can be used to prove construct morphisms and prove, when applicable that they are isomorphisms. Thus if $F : \mathcal{A} \to \mathcal{B}$ is a left exact functor between abelian categories, the question then arises as to whether the right derived functors $R^n F : \mathcal{A} \to B$ can be characterized by some universal property. The answer is that the entire collection $\{R^n F\}_{n \geq 0}$ can be so characterized.

Fix abelian categories $\mathcal{A}$ and $\mathcal{B}$. A *cohomological ∂-functor* from $\mathcal{A}$ to $\mathcal{B}$ is the following data:

- a collection $S^n : \mathcal{A} \to \mathcal{B}$ of additive functors for all $n \geq 0$, and
- for every short exact sequence

  $$0 \to A \to B \to C \to 0$$

  in $\mathcal{A}$, a morphism

  $$\partial : S^n(C) \to S^{n+1}(A)$$

  in $\mathcal{B}$.

This data is subject to the following conditions:

(1) For every short exact sequence

$$0 \to A \to B \to C \to 0$$

the sequence

$$\cdots \to S^n(A) \to S^n(B) \to S^n(C) \xrightarrow{\partial} S^{n+1}(A) \to \cdots$$

is exact. In particular S^0 is left exact

(2) For every commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

the diagram

$$\begin{array}{ccc} S^n(C) & \xrightarrow{\partial} & S^{n+1}(A) \\ \downarrow & & \downarrow \\ S^n(C') & \xrightarrow{\partial} & S^{n+1}(A') \end{array}$$

is commutative. We denote ∂-functors by $(S^{\cdot}, \partial)$. For example if $\mathcal{A}$ has enough injectives and $F : \mathcal{A} \to \mathcal{B}$ is left exact, $(R^n F, \partial)$ is a cohomological ∂-functor where ∂ is the Bockstein defined previously.

A *morphism* $f : (S^{\cdot}, \partial) \to (T^{\cdot}, \delta)$ of cohomological ∂-functors is a collection of morphisms $f_n : S^n \to T^n$ for all $n \geq 0$ such that for every short exact sequence

$$0 \to A \to B \to C \to 0$$

the diagram

$$\begin{array}{ccc} S^n(C) & \xrightarrow{\partial} & S^{n+1}(A) \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} \\ T^n(C) & \xrightarrow{\delta} & T^{n+1}(A) \end{array}$$

is commutative.

Homological ∂-functors from $\mathcal{A}$ to $\mathcal{B}$ are defined similarly: they are sequences of additive functors $S_n : \mathcal{A} \to \mathcal{B}$ and for every short exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$, a morphism in $\mathcal{B}$

$$\partial : S_n(C) \to S_{n-1}(A)$$

is given, such that

$$\cdots \to S_n(A) \to S_n(B) \to S_n(C) \xrightarrow{\partial} S_{n-1}(A) \to \cdots$$

is exact; in particular S_0 is right exact. Furthermore the ∂ must be compatible with diagrams of two short exact sequences, as in (2) above. It is clear that a homological ∂-functor from $\mathcal{A}$ to $\mathcal{B}$ can be identified with a cohomological functor from $\mathcal{A}$ to $\mathcal{B}^{\mathrm{op}}$, and *vice versa*. So in what follows I will mostly talk about cohomological ∂-functors. Finally, a contravariant cohomological (resp. homological) ∂-functor from $\mathcal{A}$ to $\mathcal{B}$ is a cohomological ∂-functor from $\mathcal{A}^{\mathrm{op}}$ to $\mathcal{B}$ (resp. $\mathcal{B}^{\mathrm{op}}$).

Universal ∂-functors

A ∂-functor $(S^{\cdot}, \partial)$ is *universal* if for every ∂-functor $(T^{\cdot}, \delta)$, a morphism $f_0 : S^0 \to T^0$ of functors extends uniquely to a morphism of ∂-functors $(S^{\cdot}, \partial) \to (T^{\cdot}, \delta)$. This is the universal property we have in mind.

The usual sort of argument shows that a universal ∂-functor $(S^{\cdot}, \partial)$ is determined up to canonical isomorphism by S^0: if $(S^{\cdot}, \partial)$ and $(T^{\cdot}, \delta)$ are universal ∂-functors a pair of inverse isomorphisms between S^0 and T^0 extend uniquely to morphisms $(S^{\cdot}, \partial) \to (T^{\cdot}, \delta)$ and $(T^{\cdot}, \delta) \to (S^{\cdot}, \partial)$ which are easily seen to be inverse isomorphisms.

We want a criterion for universality. An additive functor $F : \mathcal{A} \to \mathcal{B}$ between additive categories is *effaceable* if for every object A if $\mathcal{A}$ there is a monomorphism $u : A \to M$ such that $F(u) = 0$. Such a morphism is called an *effacement* of A for F. A cohomological ∂-functor $(S^{\cdot}, \partial)$ is *effaceable* if the S^n are effaceable for all $n > 0$ (note the strict inequality).

Theorem

Suppose $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $(S^\cdot, \partial)$ is a cohomological ∂-functor from $\mathcal{A}$ to $\mathcal{B}$. If $(S^\cdot, \partial)$ is effaceable it is universal.

Proof: Suppose $(T^\cdot, \delta)$ is an effaceable ∂-functor and $S^0 \to T^0$ is a morphism. We will construct morphisms $S^n \to T^n$ by induction on n, the case $n = 0$ being given. Suppose $S^n \to T^n$ has been constructed, A is an object of $\mathcal{A}$ and $u : A \to M$ is an effacement of A for S^{n+1}. Let B be a cokernel of u, so that

$$0 \to A \to M \to B \to 0$$

is exact. Since $S^\cdot$ and $T^\cdot$ are ∂-functors the rows of

$$\begin{array}{ccccccc}
S^n(M) & \longrightarrow & S^n(B) & \longrightarrow & S^{n+1}(A) & \xrightarrow{0} & S^{n+1}(M) \\
\downarrow & & \downarrow & & \vdots & & \\
T^n(M) & \longrightarrow & T^n(B) & \longrightarrow & T^{n+1}(A) & &
\end{array}$$

are exact, and we must show that the dotted arrow can be filled in. The diagram shows that $S^n(B) \to S^{n+1}(A)$ is an epimorphism whose kernel is the image of $S^n(M) \to S^n(B)$, so to define this arrow it suffices to find a morphism $S^n(B) \to T^{n+1}(A)$ whose composite with $S^n(M) \to S^n(B)$ is zero. Since the diagram of solid arrows is commutative, the composition of $S^n(B) \to T^n(B) \to T^{n+1}(A)$ works. Since $S^n(M) \to S^{n+1}(A)$ is an epimorphism, the morphism $S^{n+1}(A) \to T^{n+1}(A)$ is uniquely determined by the choice of u.

We next show that the $S^{n+1}(A) \to T^{n+1}(A)$ we have constructed is independent of the choice of monomorphism $u : A \to M$, and for variable A defines a morphism of functors $S^{n+1} \to T^{n+1}$. Both assertions are proven by the same kind of argument, so we suppose we are given effacements $u : A \to M$, $u' : A' \to M'$ for S^{n+1} and T^{n+1} respectively, and a morphism $f : A \to A'$. We will show that the diagram

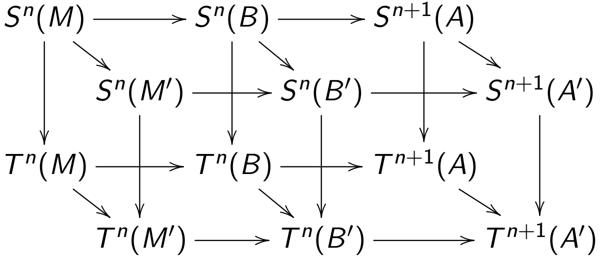
$$\begin{array}{ccc} S^{n+1}(A) & \longrightarrow & S^{n+1}(A') \\ \downarrow & & \downarrow \\ T^{n+1}(A) & \longrightarrow & T^{n+1}(A') \end{array}$$

is commutative, where the horizontal arrows are the functorial ones and the vertical arrows are determined by the choices of u and u'. When $A' = A$ and $f = 1_A$ this will show that the morphism $S^{n+1}(A) \to T^{n+1}(A)$ is independent of the choices of u and u', and then the general case will show that this morphism is functorial in A.

As before we let B and B' be the cokernels of u and u'. As a first step we consider the case where there is a morphism $g : M \to M'$ such that $gu = u'f$. Then we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{u} & M & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & A' & \xrightarrow{u'} & M' & \longrightarrow & B' & \longrightarrow & 0
\end{array}$$

where h exists by the previous argument. We now consider the diagram



in which the top and bottom rectangles yield the construction of $S^{n+1}(A) \to T^{n+1}(A)$ and $S^{n+1}(A') \to T^{n+1}(A')$. We want to show that the square at the extreme right is commutative. In the right hand cube all the other faces are commutative, the left hand face of the cube since $S^n \to T^n$ is a morphism, the top and bottom faces since S^n and T^n are ∂-functors, and the front and back faces by the construction of $S^{n+1}(A) \to T^{n+1}(A)$ and $S^{n+1}(A') \to T^{n+1}(A')$. Since $S^n(B) \to S^{n+1}(A)$ is an epimorphism the same diagram-chase we performed earlier shows that the right-hand face is commutative, which is what we wanted.

In the general case we replace the given monomorphism $u : A \to M$ by $(u, u'f) : A \to M \oplus M'$. This is a monomorphism since u is monomorphism, and on the other hand

$$S^{n+1}(u, u'f) = (S^{n+1}(u), S^{n+1}(u')S^{n+1}(f)) = (0, 0S^{n+1}(f)) = 0$$

so that $(u, u'f)$ is an effacement of A for S^{n+1}. Since the diagram

$$\begin{array}{ccc} A & \xrightarrow{(u,u'f)} & M \oplus M' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p_2} \\ A' & \xrightarrow[u']{} & M' \end{array}$$

is commutative we are reduced to the previous case.

The last step is to show the compatibility of the morphisms $S^n \to T^n$ are compatible with the connecting homomorphisms. Suppose

$$0 \to A \xrightarrow{i} B \to C \to 0.$$

is an exact sequence in $\mathcal{A}$ and let $u : B \to M$ be an effacement of B for S^{n+1}. As before, since i is a monomorphism $ui : A \to M$ is an effacement of A for S^{n+1}, and we let $M \to D$ be a cokernel of $A \to M$. From the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & M & \longrightarrow & D & \longrightarrow & 0 \end{array}$$

we get a commutative square

$$\begin{array}{ccc} S^n(C) & \xrightarrow{\partial} & S^{n+1}(A) \\ \downarrow & & \| \\ S^n(D) & \xrightarrow{\partial} & S^{n+1}(A) \end{array}$$

since $S^{\cdot}$ is a ∂-functor. On the other hand the construction of the morphism $S^{\cdot} \to T^{\cdot}$ shows that

$$\begin{array}{ccc} S^n(D) & \xrightarrow{\partial} & S^{n+1}(A) \\ \downarrow & & \downarrow \\ T^n(D) & \xrightarrow{\partial} & T^{n+1}(A) \end{array}$$

is also commutative.

Finally the original 6-diagram shows that the connecting homomorphism $T^n(C) \xrightarrow{\partial} T^{n+1}(A)$ factors

$$T^n(C) \to T^n(D) \xrightarrow{\partial} T^{n+1}(A)$$

So combining the last two squares yields a commutative square

$$\begin{array}{ccc} S^n(C) & \xrightarrow{\partial} & S^{n+1}(A) \\ \downarrow & & \downarrow \\ T^n(C) & \xrightarrow{\partial} & T^{n+1}(A) \end{array}$$

which is what we wanted to prove. ■

Corollary

Suppose $\mathcal{A}$ is an abelian category with enough injectives and $F : \mathcal{A} \to \mathcal{B}$ is a left exact functor. The sequence of right derived functors $R^{\cdot}F$ with their natural connecting morphisms is the unique universal cohomological ∂-functor from $\mathcal{A}$ to $\mathcal{B}$ whose degree 0 component is isomorphic to F.

Proof: It suffices to show that R^nF is effaceable for $n > 0$. If A is any object of $\mathcal{A}$ there is a monomorphism $u : A \to I$ in $\mathcal{A}$ for some injective object I, and we have seen that $R^nF(I) = 0$ for all $n > 0$ and *any* injective I. In particular $R^nF(u) = 0$, so R^nF is effaceable. ∎

The dual argument (or the same argument with $\mathcal{B}^{\mathrm{op}}$ replacing $\mathcal{B}$ shows that if $\mathcal{A}$ has enough projectives the left derived functors $(L_{\cdot}, \partial)$ are a universal homological ∂-functor from $\mathcal{A}$ to $\mathcal{B}$.

In particular if $\mathcal{A}$ has enough injectives, $(R^{\cdot}F, \partial)$ is up to canonical isomorphism the unique universal cohomological ∂-functor from $\mathcal{A}$ to $\mathcal{B}$ whose degree 0 component is isomorphic to F. Similarly for the left derived functors if $\mathcal{A}$ has enough projectives.

We now apply this theory in one of the more important cases. Suppose $\mathcal{A}$ has enough injectives and M is an object of $\mathcal{A}$. We have remarked that the functor

$$N \mapsto \mathrm{Hom}_{\mathcal{A}}(M, N)$$

is left exact, so it has right derived functors which we denote by $\mathrm{Ext}^n_{\mathcal{A}}(M, N)$. By construction $\mathrm{Ext}^n_{\mathcal{A}}(M, N)$ is functorial in N; that it is also functorial in M as well follows from our theorem on universal ∂-functors, together with the theorem that right derived functors are universal. In fact a morphism $f : M \to M'$ in $\mathcal{A}$ induces a morphism $\mathrm{Hom}_{\mathcal{A}}(M', \_) \to \mathrm{Hom}_{\mathcal{A}}(M, \_)$ of functors; since the right derived functors $\mathrm{Ext}^n_{\mathcal{A}}(M', \_)$ form a universal cohomological ∂-functor, the morphism $\mathrm{Hom}_{\mathcal{A}}(M', \_) \to \mathrm{Hom}_{\mathcal{A}}(M, \_)$ extends uniquely to a morphism of cohomological ∂-functors $\mathrm{Ext}^n_{\mathcal{A}}(M', \_) \to \mathrm{Ext}^n_{\mathcal{A}}(M, \_)$.

Concretely this means the following:

- For any pair of morphisms $M \to M'$, $N \to N'$ there is a commutative square

$$\begin{array}{ccc} \mathrm{Ext}^n_{\mathcal{A}}(M', N) & \longrightarrow & Ext^n_{\mathcal{A}}(M, N) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^n_{\mathcal{A}}(M', N') & \longrightarrow & Ext^n_{\mathcal{A}}(M, N') \end{array}$$

- For any short exact sequence

$$0 \to N' \to N \to N'' \to 0$$

there is a long exact sequence

$$\cdots \to \mathrm{Ext}^n_{\mathcal{A}}(M, N') \to \mathrm{Ext}^n_{\mathcal{A}}(M, N) \to$$
$$\to \mathrm{Ext}^n_{\mathcal{A}}(M, N'') \xrightarrow{\partial} \mathrm{Ext}^{n+1}_{\mathcal{A}}(M, N') \to \cdots$$

- For any commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P' & \longrightarrow & P & \longrightarrow & P'' & \longrightarrow & 0 \end{array}$$

  the diagram

$$\begin{array}{ccc} \operatorname{Ext}^n_{\mathcal{A}}(M, N'') & \xrightarrow{\partial} & \operatorname{Ext}^n_{\mathcal{A}}(M, N') \\ \downarrow & & \downarrow \\ \operatorname{Ext}^n_{\mathcal{A}}(M, P'') & \xrightarrow{\partial} & \operatorname{Ext}^n_{\mathcal{A}}(M, P') \end{array}$$

  is commutative.

We will discuss the significance of the $\mathrm{Ext}^n_{\mathcal{A}}$ later. For now let us raise the following question: if $\mathcal{A}$ has enough projectives, the contravariant functor $\mathrm{Hom}_{\mathcal{A}}(M, N)$ is additive in its first variable M and thus has right derived functors. If $\mathcal{A}$ has enough projectives *and* enough injectives, how are these two series of derived functors related?

For the moment let's denote by $\tilde{\mathrm{Ext}}^n_{\mathcal{A}}$ the derived functors of $\mathrm{Hom}_{\mathcal{A}}$ with respect to its *first* variable. Arguments parallel to the previous ones show that

- the objects $\tilde{\mathrm{Ext}}^n_{\mathcal{A}}(M, N)$ are also functorial in N, and
- for any short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

  the long exact sequence of right derived functors is

$$\cdots \to \tilde{\mathrm{Ext}}^n_{\mathcal{A}}(M'', N) \to \tilde{\mathrm{Ext}}^n_{\mathcal{A}}(M, N) \to$$
$$\to \tilde{\mathrm{Ext}}^n_{\mathcal{A}}(M', N) \xrightarrow{\partial} \tilde{\mathrm{Ext}}^{n+1}_{\mathcal{A}}(M'', N) \to \cdots$$

We will show that in fact $\operatorname{Ext}^n_{\mathcal{A}} \simeq \tilde{\operatorname{Ext}}^n_{\mathcal{A}}$ as bifunctors $\mathcal{A} \times \mathcal{A} \to B$, so the previous exact sequence can be interpreted as a long exact sequence

$$\cdots \to \operatorname{Ext}^n_{\mathcal{A}}(M'', N) \to \operatorname{Ext}^n_{\mathcal{A}}(M, N) \to$$
$$\to \operatorname{Ext}^n_{\mathcal{A}}(M', N) \xrightarrow{\partial} \operatorname{Ext}^{n+1}_{\mathcal{A}}(M'', N) \to \cdots$$

In fact we shall give two proofs of this, a simple, abstract and completely mysterious one and a completely explicit, annoying one. Today we do the first one, saving the other for later.

The first works by giving the $\tilde{\operatorname{Ext}}^n_{\mathcal{A}}(M, N)$ the structure of an structure of a cohomological ∂-functor of N, and then showing that this ∂-functor is universal. Since its degree 0 component is $\tilde{\operatorname{Ext}}^n_{\mathcal{A}}(M, N) \simeq \operatorname{Hom}_{\mathcal{A}}(M, N)$, this will show that $\tilde{\operatorname{Ext}}^n_{\mathcal{A}}(M, N) \simeq \operatorname{Ext}^n_{\mathcal{A}}(M, N)$ since the $\operatorname{Ext}^n_{\mathcal{A}}(M, N)$ are the derived functors of $\operatorname{Hom}_{\mathcal{A}}(M, N)$ in the variable N.

For any given M the $\tilde{\text{Ext}}^n_{\mathcal{A}}(M, N)$ are computed as follows: choose a projective resolution $P_\cdot \to M$ of M; then

$$\tilde{\text{Ext}}^n_{\mathcal{A}}(M, N) = H^n(\text{Hom}_{\mathcal{A}}(P_\cdot, N))$$

by definition. Suppose that

$$0 \to N' \to N \to N'' \to 0$$

is an exact sequence in $\mathcal{A}$. Since each P_i is projective the sequence

$$0 \to \text{Hom}_{\mathcal{A}}(P_i, N') \to \text{Hom}_{\mathcal{A}}(P_i, N) \to \text{Hom}_{\mathcal{A}}(P_i, N'') \to 0$$

is exact. Since $\text{Hom}_{\mathcal{A}}$ is functorial in its first argument, these sequences glue together to form a short exact sequence

$$0 \to \text{Hom}_{\mathcal{A}}(P_\cdot, N') \to \text{Hom}_{\mathcal{A}}(P_\cdot, N) \to \text{Hom}_{\mathcal{A}}(P_\cdot, N'') \to 0$$

in $C^\cdot(\mathcal{A})$.

The long exact sequence associated to this short exact sequence of complexes is

$$\to \tilde{\mathrm{Ext}}^n_{\mathcal{A}}(M, N') \to \tilde{\mathrm{Ext}}^n_{\mathcal{A}}(M, N) \to \tilde{\mathrm{Ext}}^n_{\mathcal{A}}(M, N'') \xrightarrow{\partial} \tilde{\mathrm{Ext}}^n_{\mathcal{A}}(M, N') \to$$

To complete the construction we must show that if

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L' & \longrightarrow & L & \longrightarrow & L'' & \longrightarrow & 0 \end{array}$$

is commutative with exact rows the diagram

$$\begin{array}{ccc} \tilde{\mathrm{Ext}}^n_{\mathcal{A}}(M, N'') & \xrightarrow{\partial} & \tilde{\mathrm{Ext}}^n_{\mathcal{A}}(M, N') \\ \downarrow & & \downarrow \\ \tilde{\mathrm{Ext}}^n_{\mathcal{A}}(M, L'') & \xrightarrow{\partial} & \tilde{\mathrm{Ext}}^n_{\mathcal{A}}(M, L') \end{array}$$

is commutative.

However the commutative 6-diagram leads to a diagram of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(P_., N') & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(P_., N) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(P_., N'') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(P_., L') & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(P_., L) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(P_., L'') & \longrightarrow & 0 \end{array}$$

the is commutative and the rows are exact. The desired commutative square then follows from the functoriality of the connecting morphism for a 6-diagram of complexes.

It remains to show that the $\tilde{\mathrm{Ext}}^n_{\mathcal{A}}(M, N)$ are effaceable for $n > 0$. For this it suffices to show that $\tilde{\mathrm{Ext}}^n_{\mathcal{A}}(M, I) = 0$ whenever $n > 0$ and I is injective. But this is clear: since I is injective and

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

is exact, so is

$$0 \to \mathrm{Hom}_{\mathcal{A}}(M, I) \to \mathrm{Hom}_{\mathcal{A}}(P_0, I) \to \mathrm{Hom}_{\mathcal{A}}(P_1, I) \to \mathrm{Hom}_{\mathcal{A}}(P_2, I) \to \cdots$$

is also exact, and this says that $H^n(\mathrm{Hom}_{\mathcal{A}}(P., I) = 0$ for $n > 0$, i.e. $\tilde{\mathrm{Ext}}^n_{\mathcal{A}}(M, I) = 0$.

From now on we will identify the functors $\mathrm{Ext}^n_{\mathcal{A}}$ and $\tilde{\mathrm{Ext}}^n_{\mathcal{A}}$, and call them the *Ext groups* of M and N. We will see that sometimes this identification can create problems, but for now we just appreciate the fact that when $\mathcal{A}$ has enough projectives and injectives (i.e. module categories) these functors satisfy *two* long exact sequences: for any short exact sequence

$$0 \to N' \to N \to N'' \to 0$$

a long exact sequence

$$\cdots \to \mathrm{Ext}^n_{\mathcal{A}}(M, N') \to \mathrm{Ext}^n_{\mathcal{A}}(M, N) \to \\ \to \mathrm{Ext}^n_{\mathcal{A}}(M, N'') \xrightarrow{\partial} \mathrm{Ext}^{n+1}_{\mathcal{A}}(M, N') \to \cdots$$

for any M, and for any short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

a long exact sequence

$$\cdots \to \mathrm{Ext}^n_{\mathcal{A}}(M'', N) \to \mathrm{Ext}^n_{\mathcal{A}}(M, N) \to$$
$$\to \mathrm{Ext}^n_{\mathcal{A}}(M', N) \xrightarrow{\partial} \mathrm{Ext}^{n+1}_{\mathcal{A}}(M'', N) \to \cdots$$

for any N.