Homological Algebra
Lecture 9

Richard Crew

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Interpretation of the Ext groups

Now that we have defined the Ext groups $\mathrm{Ext}^n_{\mathcal{A}}(M, N)$ for any abelian category with enough projectives or injectives, we ask what they mean. I will work out the case where $\mathcal{A}$ has enough injectives and the Ext groups are the derived functors of $\mathrm{Hom}_{\mathcal{A}}$ in the second variable, since this is the more useful case in general, although for module categories the other construction is more useful. In any case it is dual to the first.

We start with the case $n = 1$. For any objects M, N of $\mathcal{A}$, an *extension of M by N* is a short exact sequence

$$0 \to N \to E \to M \to 0$$

in $\mathcal{A}$. A *morphism* of extensions of M by N is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0\,. \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & E' & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Specifically this is a morphism from the extension in the top row to the extension in the bottom row. Note that the left and right vertical arrows must be the identity. The snake lemma shows tha $E \to E'$ must be an isomorphism in $\mathcal{A}$.

It is easily checked that extensions of M by N form a category, and from the last remark this category is actually a groupoid (all morphisms are isomorphisms). We denote it by $\mathrm{EXT}(M, N)$.

I will show that $\mathrm{EXT}(M, N)$ has a *set* of isomorphism classes and that this set is in a bijection with the group $\mathrm{Ext}^1_{\mathcal{A}}(M, N)$ constructed earlier. This suggests that the category $\mathrm{EXT}(M, N)$ must have some kind of "categorical group structure" and this is indeed the case. In what follows I will fix an extension

$$0 \to N \to E \to M \to 0 \tag{1}$$

of M by N.

Since $\mathcal{A}$ has enough injectives we can find a monomorphism $N \to I^0$ with injective I^0. Let $I^0 \to B$ be a cokernel of $N \to I^0$, so we now have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0\,. \\ & & \| & & \vdots & & & & \\ 0 & \longrightarrow & N & \longrightarrow & I^0 & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

The dotted arrow can be filled in since I^0 is injective, and then general abelian categorical nonsense shows that there is a morphism $f : M \to B$ making

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow f & & \\ 0 & \longrightarrow & N & \longrightarrow & I^0 & \longrightarrow & B & \longrightarrow & 0 \end{array} \tag{2}$$

commutative. Note that this is a pullback diagram: E is a fibered product $I^0 \times_B M$.

From this we see that the original extension (1) can be recovered up to isomorphism from the morphism $f : M \to B$. Conversely any $f : M \to B$ gives rise to an object of $\mathrm{EXT}(M, N)$ by pulling back the extension

$$0 \to N \to I^0 \to B \to 0 \tag{3}$$

by $f : M \to B$. From this we see that there is a set of isomorphism classes of objects of $\mathrm{EXT}(M, N)$, and that this set is a quotient of $\mathrm{Hom}_{\mathcal{A}}(M, B)$. We now want to describe this quotient more exactly.

Before doing this let's fix an injective resolution

$$0 \to B \to I^1 \to I^2 \to \cdots$$

of B. By splicing this onto (3) we get an injective resolution

$$0 \to N \to I^0 \to I^1 \to I^2 \to \cdots$$

of N. Since $\mathrm{Hom}_{\mathcal{A}}$ is left exact,

$$0 \to \mathrm{Hom}_{\mathcal{A}}(M, B) \to \mathrm{Hom}_{\mathcal{A}}(M, I^1) \to \mathrm{Hom}_{\mathcal{A}}(M, I^2)$$

is exact and so we can identify

$$\mathrm{Hom}_{\mathcal{A}}(M, B) \simeq \mathrm{Ker}(\mathrm{Hom}_{\mathcal{A}}(M, I^1) \to \mathrm{Hom}_{\mathcal{A}}(M, I^2)).$$

On the other hand we can insert this resolution of N into the diagram (2), resulting in

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \| & & \downarrow & & \downarrow f & & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & I^0 & \xrightarrow{d} & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots \end{array} \tag{4}$$

and I have changed the meaning of f. In this diagram f must factor through the kernel $B \to I^1$ of $I^1 \to I^2$, i.e.

$$f \in \mathrm{Ker}(\mathrm{Hom}_{\mathcal{A}}(M, I^1) \to \mathrm{Hom}_{\mathcal{A}}(M, I^2)).$$

On the other hand f was determined by the original choice of $E \to I^0$, and there are many possibilities for this. However all of them correspond to morphisms of the top complex to the bottom one, and since the bottom one is an injective resolution of N, all such morphisms are homotopic.

What this means in the current setting is this: if f and f' are morphisms making (4) a morphism of complexes, there is a morphism $h : M \to I^0$ such that

$$f' - f = dg$$

or in other words f' and f are congruent modulo the image of

$$\mathrm{Hom}_{\mathcal{A}}(M, I^0) \to \mathrm{Hom}_{\mathcal{A}}(M, I^1).$$

In other words the image of f in $H^1(\mathrm{Hom}_{\mathcal{A}}(M, I^\cdot))$ is well-defined, and the original extension (1) can be reconstructed from it up to isomorphism. Conversely if we are given an element of $H^1(\mathrm{Hom}_{\mathcal{A}}(M, I^\cdot))$ we can lift it to a element of the kernel of $\mathrm{Hom}_{\mathcal{A}}(M, I^1) \to \mathrm{Hom}_{\mathcal{A}}(M, I^2)$, and the pullback construction yields an object of $\mathrm{EXT}(M, N)$. These operations are clearly mutually inverse, so we have proven the following:

Theorem

Suppose $\mathcal{A}$ has enough injectives. For any two objects M and N of $\mathcal{A}$ the preceding constructions yield a bijection of $\operatorname{Ext}^1_{\mathcal{A}}(M,N)$ with the set of isomorphism classes of $\operatorname{EXT}(M,N)$.

When $\mathcal{A}$ has enough projectives we can use the dual procedure to again identify the isomorphism classes of $\operatorname{EXT}(M,N)$ with $\operatorname{Ext}^1_{\mathcal{A}}(M,N)$. In this case we choose a projective resolution $P. \to M$ of M. Then there is a morphism of complexes

$$
\begin{array}{ccccccccccc}
\longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\
\longrightarrow & 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0
\end{array}
$$

where f is now an element of the kernel of $\operatorname{Hom}_{\mathcal{A}}(P_1,N) \to \operatorname{Hom}_{\mathcal{A}}(P_2,N)$ and is well-defined modulo the image of $\operatorname{Hom}_{\mathcal{A}}(P_0,N) \to \operatorname{Hom}_{\mathcal{A}}(P_1,N)$.

This argument shows that the isomorphism classes of objects of $\mathrm{EXT}(M, N)$ is now in a bijection with the elements of $H^1(\mathrm{Hom}_{\mathcal{A}}(P., N))$, which is isomorphic to $\mathrm{Ext}^1_{\mathcal{A}}(M, N)$ has enough projectives *and* enough injectives. But now the following problem arises: is the identification of $H^1(\mathrm{Hom}_{\mathcal{A}}(M, I^{\cdot}))$ with $H^1(\mathrm{Hom}_{\mathcal{A}}(P., N))$ consistent with the classes we have associated to the given extension (1)? The answer is anything but clear since we don't really have an explicit formula for what this identification was, it just emerges from some universal property. We will deal with this later.

The groups $\mathrm{Ext}^n_{\mathcal{A}}(M, N)$ can be interpreted in a similar way, although the result is not quite so easy to state. Let's recall that an n-extension of M by N is an exact sequence

$$0 \to N \to E^0 \to E^1 \to \cdots \to E^{n-1} \to M \to 0 \tag{5}$$

of length $n + 2$. When $n = 1$ we recover the previous notion of an extension of M by N, which is now a 1-extension of M by N.

We can associate to the n-extension (5) an element of $\mathrm{Ext}^n_{\mathcal{A}}(M, N)$ in pretty much the same way as before: choose an injective resolution $N \to I^{\cdot}$ of N; by our earlier results on resolutions there is a morphism of complexes

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & N & \longrightarrow & E^0 & \longrightarrow & E^1 & \longrightarrow & \cdots & \longrightarrow & E^{n-1} & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow f & & \downarrow \\
0 & \longrightarrow & N & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots & \longrightarrow & I^{n-1} & \longrightarrow & I^n & \longrightarrow & I^{n+1}
\end{array}$$

and $E^{\cdot} \to I^{\cdot}$ is unique up to homotopy. Then f is an element of the kernel of $\mathrm{Hom}_{\mathcal{A}}(M, I^n) \to \mathrm{Hom}_{\mathcal{A}}(M, I^{n+1})$ and is unique modulo the image of $\mathrm{Hom}_{\mathcal{A}}(M, I^{n-1}) \to \mathrm{Hom}_{\mathcal{A}}(M, I^n)$. The resulting element of $H^n(\mathrm{Hom}_{\mathcal{A}}(M, I^{\cdot}) = \mathrm{Ext}^n_{\mathcal{A}}(M, N)$ is the *class* of the extension (5).

It's easy to see that every element of $\mathrm{Ext}^n_{\mathcal{A}}(M, N)$ arises in this way. For $k \geq 0$ let $Z^k = \mathrm{Ker}(I^k \to I^{k+1})$. Then the bottom row of the last row can be broken up into two exact sequences

$$0 \to N \to I^0 \to I^1 \to \cdots \to I^{n-2} \to Z^{n-1} \to 0$$

$$0 \to Z^{n-1} \to I^{n-1} \to Z^n \to 0$$

and the morphism $M \to I^n$ factors uniquely through a morphism $M \to Z^n$. We can pull back the second exact sequence above by this morphism to get a short exact sequence

$$0 \to Z^{n-1} \to E \to M \to 0$$

and this can be spliced onto the first one to get an n-extension

$$0 \to N \to I^0 \to I^1 \to \cdots \to I^{n-2} \to E \to M \to 0$$

whose class is that of the image of $M \to Z^n$ in $\mathrm{Ext}^n_{\mathcal{A}}(M, N)$.

Less obvious is exactly what kind of categorical equivalence relation should be put on the category of n-extensions so that two n-extensions are equivalent if and only if they have the same class. We will deal with this later (maybe).

Tor functors

Let A be a ring with identity (not necessarily commutative). The category $\mathbf{Mod}_A$ of left A-modules is an abelian category, in fact a particularly nice one: it has arbitrary limits and colimits, a generator (the ring A itself considered as a left A-module) and it has enough projectives and injectives. In particular we can compute Ext groups using either a projective resolution of the first argument or an injective resolutions of the second. The same goes for the category of right A-modules, which is equivalent to the category of left A^{op}-modules.

Suppose now that M is a fixed *right* A-module. The construction

$$N \mapsto M \otimes_A N$$

defines a right exact functor $\mathbf{Mod}_A \to \mathbf{Ab}$, and so has left derived functors which we denote by $N \mapsto \mathrm{Tor}_n^A(M, N)$. They are computed by choosing a projective resolution $P. \to N$ of N, and then

$$\mathrm{Tor}_n^A(M, N) = H_n(M \otimes_A P.).$$

If

$$0 \to N' \to N \to N'' \to 0$$

is an exact sequence the long exact sequence of left derived functors is

$$\cdots \to \mathrm{Tor}_n^A(M, N') \to \mathrm{Tor}_n^A(M, N) \to \mathrm{Tor}_n^A(M, N'') \to \mathrm{Tor}_{n-1}^A(M, N') \to \cdots$$

Furthermore $\mathrm{Tor}_n^A(M, N)$ is functorial in *both* of its first argument. Suppose $f : M \to M'$ is an A-module homomorphism. The functors $\mathrm{Tor}^n_A(M, \_)$ and $\mathrm{Tor}^n_A(M', \_)$ are both universal homological ∂-functors, so $f \otimes_A 1 : M \otimes_A \_ \to M' \otimes_A \_$ is the degree 0 part of a morphism of homological ∂-functors $(\mathrm{Tor}^n_A(M, N), \partial) \to (\mathrm{Tor}^n_A(M', N), \partial)$. In particular we get morphisms of functors $\mathrm{Tor}^n_A(M, \_) \to \mathrm{Tor}^n_A(M', \_)$ for all $n \geq 0$, which for $n = 0$ is $f \otimes_A 1$.

On the other hand if N is a fixed left A-module,

$$M \mapsto \ M \otimes_A N$$

is a right exact functor $\mathbf{Mod}_{A^{\mathrm{op}}} \to \mathbf{Ab}$, so it also has left derived functors, which for the moment we will denote by $M \mapsto \tilde{\mathrm{Tor}}_n^A(M, N)$. They are computed by choosing a projective resolution $Q_{\cdot} \to M$ of M, and then

$$\tilde{\mathrm{Tor}}_n^A(M, N) = H_n(Q_{\cdot} \otimes_A N).$$

We can use the same sort of arguments used in the last lecture show that $\tilde{\mathrm{Tor}}_n^A(M, N) \simeq \mathrm{Tor}_n^A(M, N)$, in fact functorially in both arguments. However the tensor product is a right exact functor, so instead of using cohomological ∂-functors, as for the Ext groups we must use homological ∂-functors. A homological ∂-functor $\mathcal{A} \to \mathcal{B}$ is a cohomological ∂-functor $\mathcal{A} \to \mathcal{B}^{\mathrm{op}}$, so our earlier results on universality and effaceability for cohomological ∂-functors are also valid for homological ∂-functors. However the terminology is a little different, so I will state them explicitly.

A homological ∂-functor $(S., d)$ from $\mathcal{A}$ to $\mathcal{B}$ is *universal* if for every homological ∂-functor $(T., \delta)$, a morphism $f_0 : T_0 \to S_0$ extends uniquely to a morphism of ∂-functors $f : (T., \delta) \to (S., \partial)$. If $(S., d)$ and $(T., \delta)$ are universal homological ∂-functors, an isomorphism $f_0 : S_0 \to T_0$ extends uniquely to an isomorphism $f : (T., \delta) \to (S., \partial)$ of homological ∂-functors.

An additive functor $F : \mathcal{A} \to \mathcal{B}$ is *coeffaceable* if for every object A of $\mathcal{A}$ there is an epimorphism $u : P \to A$ such that $F(u) = 0$. Note that a coeffaceable functor $F : \mathcal{A} \to \mathcal{B}$ is the same as an effaceable contravariant functor from $\mathcal{A}^{\mathrm{op}}$ to $\mathcal{B}$. A homological ∂-functor $(S., d)$ is coeffaceable if S_n is effaceable for all $n > 0$. The analogue of the earlier results on universality are as follows:

- A coeffaceable homological ∂-functor from $\mathcal{A}$ to $\mathcal{B}$ is universal.
- If $\mathcal{A}$ has enough projectives and $F : \mathcal{A} \to \mathcal{B}$ is right exact, $(L_n F, \partial)$ is the unique universal homological ∂-functor from $\mathcal{A}$ to $\mathcal{B}$ whose degree 0 component is isomorphic to F.

Let's return to the case $\mathcal{A} = \mathbf{Mod}_A$. To show that $\tilde{\text{Tor}}_n^A(M, N) \simeq \text{Tor}_n^A(M, N)$ it suffices to show that the sequence of $\tilde{\text{Tor}}_n^A(M, \_)$ for $n \geq 0$ has the structure of a universal homological ∂-functor. Since

$$\tilde{\text{Tor}}_0^A(M, N) \simeq M \otimes_A N \simeq \text{Tor}_0^A(M, N)$$

the universal properties of $\text{Tor}_n^A(M, N)$ and $\tilde{\text{Tor}}_n^A(M, N)$ will yield an isomorphism $(\text{Tor}_n^A(M, N), \partial) \xrightarrow{\sim} (\tilde{\text{Tor}}_n^A(M, N), \partial)$ of homological ∂-functors, and in particular isomorphisms $\text{Tor}_n^A(M, N) \simeq \tilde{\text{Tor}}_n^A(M, N)$ functorial in N. Functoriality in M can be shown by giving the $\text{Tor}_n^A(\_, N)$ the structure of a universal homological ∂-functor and arguing as before to get an isomorphism $\text{Tor}_n^A(M, N) \simeq \tilde{\text{Tor}}_n^A(M, N)$ that is functorial in M. One must then show that the two isomorphisms so obtained are in fact the same, which I will leave as an exercise.

Suppose first that

$$0 \to N' \to N \to N'' \to 0$$

is an exact sequence of left A-modules and $P. \to M$ is a projective resolution. Since a projective module is flat,

$$0 \to P_i \otimes_A N' \to P_i \otimes_A N \to P_i \otimes_A N'' \to 0$$

is exact for all i, and therefore

$$0 \to P. \otimes_A N' \to P. \otimes_A N \to P. \otimes_A N'' \to 0$$

is an exact sequence of complexes. The long exact sequence of homology

$$\cdots \to H_n(P.\otimes_A N') \to H_n(P.\otimes_A N) \to H_n(P.\otimes_A N'') \to H_{n-1}(P.\otimes_A N') \to \cdots$$

is

$$\cdots \to \tilde{\mathrm{Tor}}_n^A(M, N') \to \tilde{\mathrm{Tor}}_n^A(M, N) \to \tilde{\mathrm{Tor}}_n^A(M, N'') \xrightarrow{\partial} \tilde{\mathrm{Tor}}_n^A(M, N'') \to \cdots$$

and it is easily checked that $(\tilde{\mathrm{Tor}}_n^A(M, \_), \partial)$ is a homological ∂-functor.

I will show that $(\tilde{\mathrm{Tor}}_n^A(M, \_), \partial)$ is universal by showing that it is coeffaceable. In fact for any left A-module N there is an epimorphism $u : P \to N$ with P projective, so it suffices to show that $\tilde{\mathrm{Tor}}_n^A(M, P) = 0$ when $n > 0$ and P is projective. If $Q_\cdot \to M$ is a projective resolution,

$$\cdots \to Q_2 \to Q_1 \to Q_0 \to M \to 0$$

is exact by definition and therefore

$$\cdots \to Q_2 \otimes_A P \to Q_1 \otimes_A P \to Q_0 \otimes_A P \to M \otimes_A P \to 0$$

is exact since P is flat. This shows that $\tilde{\mathrm{Tor}}_n^A(M, P) = 0$ for $n > 0$.

Recall that a right A-module M is *flat* if for every exact sequence

$$0 \to N' \to N \to N'' \to 0$$

the sequence

$$0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

is exact. Similarly for left A-modules.

Proposition

For any right A-module M the following are equivalent:

1. *M is flat;*
2. $\mathrm{Tor}_n^A(M, N) = 0$ *for all left A-modules N and all $n > 0$;*
3. $\mathrm{Tor}_1^A(M, N) = 0$ *for all left A-modules N.*

Proof: (1) implies (2): If $P. \to N$ is a projective resolution of N, $\mathrm{Tor}_n^A(M, N) = H_n(M \otimes_A P.)$ by definition. Now $P.$ is exact in positive degrees (i.e. $H_n(P.) = 0$ for $n > 0$ and since M is flat, $M \otimes_A P.$ is also exact in positive degrees. This says that $\mathrm{Tor}_n^A(M, N) = 0$ for $n > 0$.

(2) trivially implies (3). Suppose (3) holds; if

$$0 \to N' \to N \to N'' \to 0$$

is exact the long exact sequence of Tor is

$$0 = \mathrm{Tor}_1^A(M, N') \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

or in other words

$$0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

is exact. ∎

Corollary

Suppose

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of right A-modules.

1. *If M' and M'' are flat, so is M.*
2. *If M and M'' are flat, so is M'.*

Proof: This follows from the proposition and the long exact sequence of Tor, part of which is

$$\mathrm{Tor}_2^A(M'', N) \to \mathrm{Tor}_1^A(M', N) \to \mathrm{Tor}_1^A(M, N) \to \mathrm{Tor}_1^A(M'', N)$$

for any left A-module N. ■

It can happen that M' and M are flat without M'' being flat.

Lemma

For any functor $i \mapsto N_i$ from a filtered indexing category to $\mathbf{Mod}_A$, the natural morphism

$$\varinjlim_i \mathrm{Tor}_n(M, N_i) \to \mathrm{Tor}_n(M, \varinjlim_i N_i)$$

is an isomorphism.

Proof: The morphism is the one induced from the morphisms $\mathrm{Tor}_n(M, N_i) \to \mathrm{Tor}_n(M, \varinjlim_i N_i)$ by the universal property of colimits. By our earlier discussion we can calculate the Tor_n by choosing a projective resolution $Q_. \to M$. The canonical morphisms $N_i \to \varinjlim_i N_i$ yield morphisms

$$Q_. \otimes_A N_i \to Q_. \otimes_A \varinjlim_i N_i$$

and thus a morphism

$$\varinjlim_i (Q_\cdot \otimes_A N_i) \to Q_\cdot \otimes_A \varinjlim_i N_i$$

which is an isomorphism since tensor products commute with colimits. This yields isomorphisms

$$H_n(\varinjlim_i (Q_\cdot \otimes_A N_i)) \xrightarrow{\sim} H_n(Q_\cdot \otimes_A \varinjlim_i N_i) \xrightarrow{\sim} \mathrm{Tor}_n^A(M, \varinjlim_i N_i)$$

for all n. Finally H_n commutes with filtered colimits, so we get an isomorphism

$$\varinjlim_i \mathrm{Tor}_n^A(M, N_i) \xrightarrow{\sim} \varinjlim_i H_n(Q_\cdot \otimes_A N_i) \xrightarrow{\sim} \mathrm{Tor}_n^A(M, \varinjlim_i N_i)$$

as asserted. ∎

Corollary

A right A-module M is flat if and only if $\mathrm{Tor}_1^A(M,N)=0$ for all finitely generated left A-modules N.

Proof: Any left A-module is a filtered colimit of finitely generated ones. ∎

By using this corollary one can show that M is flat if and only if one of the following conditions holds:

- $\mathrm{Tor}_1^A(M,A/I)=0$ for any finitely generated left ideal $I\subseteq A$;
- the natural map $M\otimes_A I\to MI$ is an isomorphism for all finitely generated left ideals $I\subseteq A$.

Finally all of the above results are valid if we systematically exchange "right" and "left."