MAS 5312 – Lecture for March 18, 2020

Richard Crew

Determinants

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Now that we know about permutation groups we can give a *direct* definition of the determinant of an $n \times n$ matrix $A = (a_{ij}) \in M_{nn}(R)$:

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{1 \leq i \leq n} a_{i\sigma(i)}. \tag{1}$$

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- Multiply the results and then multiply by the sign of σ.
- Add the results. Voilà!

Note that there are $n!$ terms in the sum, so this is not the way to compute these things.

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which says that

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This means that anything we prove about columns in a determinant also applies to sums.

It will be convenient to regard a square matrix as a row vector of column vectors, in which case it is a function

$$R^n \times R^n \to R \qquad (v_1 \cdots v_n) \mapsto \det(v_1 \cdots v_n).$$

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If $M_1, \ldots, M_n, N$ are R-modules, a function

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is *multilinear* if is linear in each variable separately. It is then clear from the definition 1 that when viewed in this way, the determinant is a multilinear function $R^n \times R^n \to R$, when viewed either as a function of the columns, or of the rows of a matrix.

Proposition

If two rows or two columns of A are identical, $\det(A) = 0$.

Proof. It suffices to prove this for rows. Suppose rows r and s of A are identical, and let $\tau = (r\ s)$ and $H = \langle \tau \rangle$. If S is any choice of left coset representatives in S_n for H,

$$\begin{aligned}
\det(A) &= \sum_{\sigma \in S} \operatorname{sgn}(\sigma) \prod_{1 \le i \le n} a_{i\sigma(i)} + \sum_{\sigma \in S} \operatorname{sgn}(\sigma\tau) \prod_{1 \le i \le n} a_{i\sigma\tau(i)} \\
&= \sum_{\sigma \in S} \operatorname{sgn}(\sigma) \prod_{1 \le i \le n} a_{i\sigma(i)} - \sum_{\sigma \in S} \operatorname{sgn}(\sigma) \prod_{1 \le i \le n} a_{i\sigma\tau(i)} \\
&= \sum_{\sigma \in S} \operatorname{sgn}(\sigma) \prod_{1 \le i \le n} a_{i\sigma(i)} - \sum_{\sigma \in S} \operatorname{sgn}(\sigma) \prod_{1 \le i \le n} a_{\tau(i)\sigma(i)}
\end{aligned}$$

where for the last equality we re-index $i \mapsto \tau(i)$ and use $\tau^2 = 1$. Since $\tau(i) = i$ for $i \neq r, s$ and $a_{ri} = a_{si}$, the two sums cancel. ∎

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An alternating map is antisymmetric: for example if $F : V \times V \to W$ is alternating, $F(v, v) = 0$ for all $v \in V$, and applying this to v, w and $v + w$ yields

$$\begin{aligned} 0 &= F(v + w, v + w) \\ &= F(v, v) + F(v, w) + F(w, v) + F(w, w) \\ &= F(v, w) + F(w, v). \end{aligned}$$

If F is antisymmetric *and* 2 *is invertible in* R, F is alternating:

$$F(v,v) = -F(v,v) \implies 2F(v,v) = 0 \implies F(v,v)$$

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In fact it is close to being the *only* function with these properties, as we will see later.In any case, we have shown that if we switch two rows or columns of a matrix, the determinant changes sign. More generally, if we perform a permutation σ on the rows or columns, the determinant changes by the sign $\mathrm{sgn}(\sigma)$ of the permutation.

Since the determininant is an alternating function of the columns of a matrix, we find:

Corollary

(Cramer's Rule). Suppose

$$w = \sum_i x_i v_i.$$

Then

$$\det(v_1 \cdots v_{i-1} \; w \; v_{i+1} \cdots v_n) = x_i \det(v_1 \cdots v_n).$$

In particular if $\det(v_1 \cdots v_n) \in R^\times$ *then*

$$x_i = \frac{\det(v_1 \cdots v_{i-1} \; w \; v_{i+1} \cdots v_n)}{\det(v_1 \cdots v_n)}.$$

Applying Cramer's rule to $w = 0$, we find

Corollary

If $v_1, \ldots, v_n \in R^n$ *and* $\det(v_1 \cdots v_n) \in R^\times$, $v_1, \ldots, v_n$ *are independent.* .

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The converse is false, even if R is a domain. For example $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \end{pmatrix} \in \mathbb{Z}^2$ are independent, but $\det \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \notin \mathbb{Z}^\times$.

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Recall that the (i,j)-minor A_{ij} of a square matrix A is the matrix obtained by expunging the ith row and jth column of A, and the (i,j)-cofactor is the quantity $C_{ij} = (-1)^{i+j} \det(A_{ij})$.

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Proposition

(Cofactor expansion). For any square matrix $A = (a_{ij})$,

$$\det(A) = \sum_i a_{ij} C_{ij} = \sum_j a_{ij} C_{ij}.$$

We will just prove the first equality since the second follows on applying it to A^T.

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Lemma

If

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & * \\ 0 & B \end{pmatrix}$$

in block form, where $a \in R$ and $B \in M_{n-1,n-1}(R)$ then $\det(A) = a \det(B)$*.*

Proof. This follows from the definition, since $a_{1j} = 0$ for $j > 1$ and thus the only $\sigma \in S_n$ that contribute to the sum are those for which $\sigma(j) \geq j$ for $j > 1$, i.e. $\sigma \in S_{n-1}$. ∎

The proposition follows from the multilinearity of the determinant and from:

Corollary

Suppose

$$A = \begin{pmatrix} A_1 & 0 & A_2 \\ * & a_{ij} & * \\ A_3 & 0 & A_4 \end{pmatrix}$$

in block form, where the entry a_{ij} *lies in the* (i, j) *position. Then* $\det(A) = a_{ij}C_{ij}$ *where* C_{ij} *is the* (i, j)*-cofactor of* A*.*

Proof. We can move the a_{ij} into the $(1, 1)$-position of the matrix by applying the permutation $(1\ 2\ \cdots i)$ to the rows and $(1\ 2\ \cdots j)$ to the columns. These have signs $(-1)^{i-1}$ and $(-1)^{j-1}$ respectively, so the determinant changes by $(-1)^{i+j}$. Therefore

$$\det(A) = (-1)^{i+j} \det \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

and it suffices to observe that $\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ is the (i, j)-minor. ∎

Theorem

Suppose $F : V \times \cdots \times V \to R$ is an alternating function of n arguments, and v_i, $w_j \in V$ for $1 \leq i, j \leq n$ satisfy

$$w_j = \sum_j a_{ij} v_i.$$

If $A = (a_{ij})$ then

$$F(w_1, \ldots, w_n) = \det(A) F(v_1, \ldots, v_n).$$

Proof: By multilinearity,

$$F(w_1, \ldots, w_n) = \sum_i \prod_{j_i} a_{j_i i} F(v_{j_1}, \ldots, v_{j_n}).$$

Since F is alternating the only nonzero terms in the sum are those for which $(j_1, \ldots, j_n)$ is a permutation of $(1, \ldots, n)$. Since F is antisymmetric

$$F(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = \operatorname{sgn}(\sigma) F(v_1, \ldots, v_n)$$

and thus $F(v_1, \ldots, v_n)$ factors out of the sum, leaving

$$\begin{aligned} F(w_1, \ldots, w_n) &= F(v_1, \ldots, v_n) \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_i a_{\sigma(i)i} \\ &= F(v_1, \ldots, v_n) \det(A^T) = F(v_1, \ldots, v_n) \det(A) \end{aligned}$$

as asserted.

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as asserted.

Remark: in effect we have shown that for any free R-module M of rank n, the set of alternating maps $M^n \to R$ is a free R-module of rank one. In fact for any k the set of alternating maps $M^k \to R$ is a free R-module of rank $\binom{n}{k}$ (and in particular is 0 for $k > n$). This requires the theory of exterior products, which we cover in 6000 algebra.

Corollary

If A, $B \in M_{nn}(R)$ then

$$\det(AB) = \det(A)\det(B).$$

Proof. Write

$$A = (v_1 \cdots v_n) \qquad AB = (w_1 \cdots w_n)$$

and set $B = (b_{ij})$. Then

$$w_j = \sum_i b_{ij} v_i.$$

and the theorem yields

$$\det(w_1 \cdots w_n) = \det(B)\det(v_1 \cdots v_n).$$

This says that $\det(AB) = \det(B)\det(A) = \det(A)\det(B)$ as asserted. ∎

Theorem

A square matrix $A \in M_{nn}(R)$ is invertible if and only if $\det(A) \in R^\times$. If so, $\det(A^{-1}) = \det(A)^{-1}$ and

$$A^{-1} = \frac{\det(C)}{\det(A)} \tag{3}$$

where $C = (C_{ij})^T$.

If A is invertible

$$1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$$

which yields the direct implication and the formula $\det(A^{-1}) = \det(A)^{-1}$. For the converse it suffices to show that the right hand side of 3 is an inverse, so denote it by B.

Then for any i the (i,i)-entry of AB is

$$\frac{1}{\det(A)}\sum_k a_{ik}C_{ik} = 1$$

by the cofactor expansion in the ith row. On the other hand for $j \neq i$ the (i,j)-entry of AB is

$$\frac{1}{\det(A)}\sum_k a_{ik}C_{jk} = 0$$

since this is the cofactor expansion in the jth row of the determinant of the matrix obtained by replacing the jth row by the ith row. This determinant is zero since it has two identical rows.

Corollary

A set of $v_1, \ldots, v_n \in R^n$ *is a basis if and only if*

$$\det(v_1 \cdots v_n) \in R^\times$$

.

Proof. Write

$$v_j = \sum_i c_{ij} e_i$$

where as usual (e_i) is the standard basis $\mathcal{E}$ of R^n. By the theorem

$$\det(v_1, \ldots, v_n) = \det(C) \det(e_1, \ldots, e_n) = \det(C).$$

Now if $v_1, \ldots, v_n$ is a basis of R^n and $\mathcal{B} = (v_i)$ is the ordered basis, $C = (c_{ij})$ is the change-of-basis matrix $C_{\mathcal{E}\mathcal{B}}$. Therefore C is invertible and $\det(C) \in R^\times$. Conversely $\det(C) \in R^\times$, C is invertible and we can express each e_i as a linear combination of the v_i, so $v_1, \ldots, v_n$ spans R^n. We saw in an earlier corollary that $v_1, \ldots, v_n$ is independent as well, so it is a basis. ∎

We now have almost all we need for the efficient computation of determinants. The last bit is:

Proposition

If $A = (a_{ij})$ *is upper or lower triangular,*

$$\det(A) = \prod_i a_{ii}.$$

Proof. It suffices to prove the case when A is upper triangular, so that $a_{ij} = 0$ for $j < i$. The only nonzero products in the sum formula 1 are the ones for which $\sigma(i) \geq i$ for all i, and this only happens when $\sigma = 1$. ∎

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- The matrix $E(i,j,r)$, the result of adding $r \in R$ times the jth row of the identity to the ith row.

If A is an $n \times m$ matrix with entries in R, $P(\sigma)A$ is the result of permuting the rows of A by σ, $D(r)$ is the result of multiplying the ith row of A by r, and $E(i,j)A$ is the result of adding r times the jth row of A to the ith row. If A is $m \times n$ then right multiplication by these matrices has the corresponding effect on columns. As row operations are clearly reversible, elementary matrices are invertible, and in fact

$$P(\sigma)^{-1} = P(\sigma^{-1}), \qquad D(i,r)^{-1} = D(i,r^{-1}), \qquad E(i,j,r)^{-1} = E(i,j,-$$

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Suppose K is a field and $A \in M_{nn}(K)$. The following are equivalent:

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Proof. In fact any square matrix A is row-equivalent to an upper triangular matrix. If all diagonal entries are nonzero it is row-equivalent to the identity; then $\det(A) \neq 0$ and A is invertible. If on the other hand some diagonal entries are zero, $\det(A) = 0$ and A is non invertible. Similarly for column-equivalence. ∎

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Proof. If the successive row operations correspond to elementary matrices $E_1, E_2, \ldots, E_r$ then $E_r \cdots E_2 E_1 A = I$, whence

$$A = E_1^{-1} E_2^{-1} \cdots E_r^{-1}$$

and the E_i^{-1} are also elementary. ∎

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The theorem is the basis of the standard algorithm for computing the inverse of a matrix A: form the matrix $(A\ I)$ and perform row-operations on it until the left hand side is I; then the right hand side is A^{-1}. In fact the combined effect of all the row operations is obtained by multiplying $(A\ I)$ on the left by some invertible matrix B, and $B(A\ I) = (BA\ B)$. Thus when the left hand side is I, the right hand side is A^{-1}.