Supporting Information

Appendix S1: Local stability analysis

In this appendix, we present a local stability analysis of the example model (3) in the main text:

$$\frac{dS}{dt} = G - \mu S - a_R S N_R - a_I S N_I \tag{A.1}$$

$$\frac{dN_I}{dt} = \left(a_I b_I S - m_I\right) N_I \tag{A.2}$$

$$\frac{dN_R}{dt} = \left(a_R b_R S - m_R\right) N_R - \delta p N_I N_R \,. \tag{A.3}$$

This system has four equilibria: one with the resource alone and neither consumer, one with the resource and resident and no invader, one with the resource and invader and no resident, and one with positive densities for all three. For each of these four equilibria, we evaluate the Jacobian at the equilibrium ($\hat{S}, \hat{N}_I, \hat{N}_R$):

$$J = \begin{pmatrix} -(\mu + a_R \hat{N}_R + a_I \hat{N}_I) & -a_I \hat{S} & -a_R \hat{S} \\ a_I b_I \hat{N}_I & a_I b_I \hat{S} - m_I & 0 \\ a_R b_R \hat{N}_R & -\delta p \hat{N}_R & a_R b_R \hat{S} - m_R - \delta p \hat{N}_I \end{pmatrix}. \tag{A.4}$$

Then we determine local stability by either calculating the eigenvalues (locally stable if the real part of the leading eigenvalue is negative) or the Routh-Hurwitz criteria (for a 3×3 matrix, locally stable if the trace is negative, the determinant is negative, and the product of the trace and the sum of the 2×2 principal minor determinants is less than the determinant).

First, the equilibrium with the resource alone ($\hat{N}_I = \hat{N}_R = 0$, $\hat{S} = G/\mu$) is locally unstable as long as $a_i b_i G / \mu > m_i$ for either $i=I$ or $i=R$, a typical assumption for a chemostat model (growth exceeds mortality at low densities and without the competitor present). Second, the equilibrium with no invader ($\hat{N}_I = 0$, $\hat{S} = m_R/(a_R b_R)$, $\hat{N}_R = (G - \mu\hat{S})/(a_R\hat{S})$) is always locally stable given the assumption of $S_I^* > S_R^*$, or $m_I/(a_I b_I) > m_R/(a_R b_R)$ (in the absence of predation, the resident can persist at lower resource levels than the invader and is therefore the superior competitor).

Third, the equilibrium with no resident ($\hat{N}_R = 0$, $\hat{S} = m_I/(a_I b_I)$, and $\hat{N}_I = (G - \mu\hat{S})/(a_I\hat{S})$) is locally stable if

$$\frac{a_R b_R}{a_I b_I} m_I - m_R - \delta p\left(\frac{b_I G}{m_I} - \frac{\mu}{a_I}\right) < 0. \tag{A.5}$$

Fourth, the equilibrium with all three of the resource and two competitors,

$$\hat{S} = \frac{m_I}{a_I b_I} \tag{A.6}$$

$$\hat{N}_I = \frac{1}{\delta p}\left(a_R b_R \frac{m_I}{a_I b_I} - m_R\right) \tag{A.7}$$

$$\hat{N}_R = \frac{a_I b_I}{m_I a_R}\left(G - \frac{m_I}{b_I}\left(\frac{\mu}{a_I} + \frac{1}{\delta p}\left(\frac{a_R b_R}{a_I b_I} m_I - m_R\right)\right)\right), \tag{A.8}$$

exists biologically (all at positive densities) given the assumption of $S_I^* > S_R^*$ (thus $N_I > 0$) and inequality (A.5) holds (thus $N_R > 0$); this equilibrium is always locally unstable when it exists biologically.

In summary, if inequality A.5 holds, both the equilibrium with the resident and no invader and the equilibrium with the invader and no resident are locally stable, and the equilibrium that has the resident and invader coexisting exists as a locally unstable threshold in between these two attractors. Therefore, a necessary condition for the invader to successfully invade (i.e., the system goes to the equilibrium with the invader and no resident) is that the system starts beyond this threshold given by the locally unstable equilibrium with both the invader and resident, equations A.6-A.8. Focusing on the threshold density for the invader N_I in equation A.7, the invader can potentially successfully invade when

$$N_I > \frac{1}{\delta p}\left(a_R b_R \frac{m_I}{a_I b_I} - m_R \right). \quad \text{(A.9)}$$

Substituting $S_I^* = m_I/(a_I b_I)$ and $S_R^* = m_R/(a_R b_R)$ yields inequality 6 in the main text. Although realizing this inequality may require that there is a disturbance knocking the resident to low numbers, permitting the invader to become established, this threshold does describe when the previous resident cannot successfully re-establish.

We can go beyond the specific model treated above to elaborate on the general model:

$$\frac{dN_I}{dt} = N_I F_I(S) \quad \text{(A.10)}$$

$$\frac{dN_R}{dt} = N_R F_R(S) - N_R \delta\phi(N_I) \quad \text{(A.11)}$$

$$\frac{dS}{dt} = Q(S) - \sum_{i=I,R} f_i'(S) N_i, \quad \text{(A.12)}$$

We assume that the consumption and growth rates of both species increase with resource levels, and that the extra mortality of the resident due to a higher-level consumer increases with the abundance of the inferior competitor.

If we assume that an equilibrium with both species and the resource exists, the Jacobian of the above model has the sign structure of

$$\begin{bmatrix} 0 & -c & -d \\ 0 & 0 & -e \\ a & b & q \end{bmatrix} \quad \text{(A.13)}$$

where *a*, *b*, *c*, *d*, and *e* are positive constants, whereas *q* may be either positive or negative.

The dominant eigenvalue of the linearized system corresponding to this Jacobian is given by the roots of the characteristic equation:

$$y^3 + a_3 y^2 + a_2 y + a_1 = 0 \tag{A.14}$$

where $a_3 = -q$, $a_2 = ad + be$, and $a_1 = -ace$.

From the Routh-Hurwitz conditions, a necessary condition for local stability of the equilibrium is that $a_1 > 0$, which does not hold. Based upon this model, we can conclude that, in general, there is no stable equilibrium of two species competing for a single resource, where the inferior competitor increases the direct mortality of the superior species.

We caution that this does not demonstrate that coexistence is impossible, but it does show that if there is coexistence within the framework of this model, it must be because of unstable dynamics.