(1) The differential equation \( \frac{dv}{dt} = 9.8 - 0.5v \) models the velocity of a falling object. The \(-0.5v\) term models air resistance.

(a) Sketch curves for the solution of the IVP with \( v(0) = 0 \) and \( v(0) = 50 \) on the direction field.

(b) For each of the above curves, what is \( \lim_{t \to \infty} v(t) \)?
   For both, \( \lim_{t \to \infty} v(t) \) is roughly 20 (probably 9.8*2)

(c) Find an explicit specific solution for the IVP
\[
\frac{dv}{dt} = 9.8 - 0.5v, \quad v(0) = 0.
\]

\[
\begin{align*}
\frac{dv}{dt} &= 9.8 - 0.5v \\
\frac{dv}{9.8 - 0.5v} &= dt \\
-2\ln|9.8 - 0.5v| &= t + C \\
\ln|9.8 - 0.5v| &= -t/2 + C \\
|9.8 - 0.5v| &= e^{-t/2+C} = Ce^{-t/2}
\end{align*}
\]
When \( v = 0 \) (init. cond), \(|9.8 - 0.5v| > 0\), so \(|9.8 - 0.5v| = 9.8 - 0.5v\) and we have

\[ 9.8 - 0.5v = Ce^{-t/2} \]
\[ v = Ce^{-t/2} + 9.8 \times 2 \]

When \( v(0) = 0 \), \( 0 = C + 2 \times 9.8 \) so \( v(t) = 2 \times 9.8(1 - e^{-t/2}) \)

(2) Find the general explicit solution to

\[ x \frac{dy}{dx} + 3(y + x^2) = \frac{\sin(x)}{x}. \]

\[ x \frac{dy}{dx} + 3y = \frac{\sin(x)}{x} - 3x^2 \]

This is not one of our special cases because there is a \( y \) term and \( \frac{d}{dx} x \neq 3 \). Put in standard form:

\[ \frac{dy}{dx} + \frac{3}{x} y = \frac{\sin(x)}{x^2} - 3x \]

This has integrating factor \( \mu(x) = e^{\int \frac{3}{x} dx} = e^{3 \ln|x|} = |x|^3 \). Since the sign of \( \mu \) doesn’t matter we take \( \mu(x) = x^3 \) Multiplying through:

\[ x^3 \frac{dy}{dx} + 3x^2 = x \sin(x) - 3x^4 \]
\[ \frac{d}{dx}(x^3y) = \int (x \sin(x) - 3x^4) dx \]
\[ y = x^{-3} \int (x \sin(x) - 3x^4) dx \]

Using integration by parts:

\[ y = x^{-3}(\sin(x) - x \cos(x) - \frac{3x^5}{5} + C) \]

(3) Find the general implicit solution to

\[ \frac{dy}{dx} = y \ln(x^y) + y^2 \]

This is separable.

\[ \frac{dy}{dx} = y^2 \ln(x) + y^2 = y^2(\ln(x) + 1) \]
\[ \frac{dy}{y^2} = (\ln(x) + 1) dx \]
Use integration by parts to solve $\int \ln(x) dx \ (u = \ln(x), \ dv = dx)$ and obtain:

$$-y^{-1} = x \ln(x) + C$$

(4) Solve the following initial value problem:

$$(1 + \frac{1}{1 + x^2 + 2xy + y^2})dx + (y^{-1/2} + \frac{1}{1 + x^2 + 2xy + y^2})dy = 0, \ y(0) = 0$$

We have $M = 1 + \frac{1}{1 + x^2 + 2xy + y^2}$ and $N = y^{-1/2} + \frac{1}{1 + x^2 + 2xy + y^2}$.

Call $G(x, y) = \frac{1}{1 + x^2 + 2xy + y^2}$. Then $\frac{\partial M}{\partial y} = \frac{\partial G}{\partial y}$ and $\frac{\partial N}{\partial x} = \frac{\partial G}{\partial x}$. Since $G(x, y) = G(y, x)$, evidently these are the same. You are welcome to check the derivatives if you are unclear on this. So this is exact.

Now we will find $F$ so that $M, N$ are its partial derivatives.

$$\frac{\partial F}{\partial x} = 1 + \frac{1}{1 + x^2 + 2xy + y^2}$$

$$F(x, y) = \int (1 + \frac{1}{1 + x^2 + 2xy + y^2})dx + g(y)$$

$$F(x, y) = \int (1 + \frac{1}{1 + (x + y)^2})dx + g(y)$$

$$F(x, y) = x + \arctan(x + y) + g(y)$$

Now we just need to find $g(y)$.

$$N = \frac{\partial F}{\partial y} = \frac{1}{1 + (x + y)^2}dx + g'(y)$$

$$y^{-1/2} + \frac{1}{1 + (x + y)^2}dx = \frac{1}{1 + (x + y)^2}dx + g'(y)$$

$$y^{-1/2} = g'(y)$$

$$2\sqrt{y} = g(y)$$

Then the general solution is given implicitly by

$$x + \arctan(x + y) + 2\sqrt{y} = C.$$  

When $y(0)=0$ we have $0 + \arctan(0) + 2 \times 0 = C$, so $C = 0$ and the specific solution to the IVP is

$$x + \arctan(x + y) + 2\sqrt{y} = 0.$$  

(5) Find an integrating factor for the following equation. **Do not solve the equation!!**

$$(3x^2 + y)dx + (x^2y - x)dy = 0$$
Call \( M = (3x^2 + y) \), \( N = (x^2y - x) \). If \( \frac{\partial M}{\partial y} \frac{\partial N}{\partial x} = P(x) \), then
\[
\mu = \mu(x) = e^{\int P(x) \, dx}.
\]
Otherwise, if \( \frac{\partial N}{\partial x} \frac{\partial M}{\partial y} = Q(y) \), then \( \mu = \mu(y) = e^{\int Q(y) \, dy} \).
\[
\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{1 - (2xy - 1)}{x^2y - x} = \frac{-2(2xy - 1)}{x(xy - 1)} = \frac{-2}{x} = P(x).
\]
Hence \( e^{\int P(x) \, dx} = e^{\int (-2/x) \, dx} = e^{-2\ln|x|} = x^{-2} \) is an integrating factor. You should verify this.

(6) Verify that \( \mu(x, y) = xy \) is an integrating factor for the differential equation
\[
(2 \cos(x^2y) + x^{-1}) \, dx + (xy^{-1} \cos(x^2y) + y^{-1}) \, dy = 0
\]
.

Multiply through by \( \mu(x, y) = xy \).
\[
(2xy \cos(x^2y) + y) \, dx + (x^2 \cos(x^2y) + x) \, dy = 0
\]
Call \( M = 2xy \cos(x^2y) + y \), \( N = x^2 \cos(x^2y) + x \).
\[
\frac{\partial M}{\partial y} = 2xy(-\sin(x^2y) + x^2) + 2x \cos(x^2y) + 1, \quad \text{and} \quad \frac{\partial N}{\partial x} = x^2(-\sin(x^2y)(2xy) + \cos(x^2y)(2x) + 1.
\]
These are evidently the same.