(1) [20] Find the form of a solution to
\[ y'' + 2y' + 2y = f(t) \]
where
(a) \( f(t) = t^2 e^t \)
(b) \( f(t) = e^{-t} \sin(t) \)
(c) \( f(t) = 5t e^{-t} \)
(d) \( f(t) = 2t^2 e^t - t e^{-t} \)

The roots of the characteristic polynomial \( r^2 + 2r + 2 \) are \( r = -1 \pm i \).
(a) 1 is not a root, so \( y_p(t) = (A_2t^2 + a_1 t + A_0) e^t \).
(b) \(-1 + i \) is a root so \( y_p(t) = tA_0 e^{-t} \sin(t) + tB_0 e^t \cos(t) \).
(c) \(-1 \) is not a root so \( y_p(t) = (A_1 t + A_0) e^{-t} \).
(d) By the superposition principle and (a),(c),
\[
y_p(t) = (A_2t^2 + a_1 t + A_0) e^t + (B_1 t + B_0) e^{-t}
\]

(2) [15] Solve the initial value problem
\[ y'' - 3y' + 2y = 0, \quad y(0) = 1, y'(0) = 2. \]

The characteristic polynomial \( r^2 - 3r + 2 \) has \( r = 1, 2 \) as distinct real roots. Therefore the general solution to this homogeneous equation is given by
\[ y = c_1 e^t + c_2 e^{2t}, \]
and has derivative
\[ y' = c_1 e^t + 2c_2 e^{2t} \]

Plugging in the initial conditions gives the linear system
\[
\begin{align*}
1 &= c_1 + c_2 \\
2 &= c_1 + 2c_2
\end{align*}
\]
Taking the difference of the two equations gives \(-1 = -c_2\) so \( c_2 = 1 \). Back substitution gives \( c_1 = 0 \). So the solution which solves the IVP is given by
\[ y = e^{2t} \]

(3) [20] Find the general solution to
\[ (2x + y - 1)dx + (x - y - 2)dy = 0. \]

This is an equation with linear coefficients. It has been pointed out to me that it is also exact. The (much shorter) solution of this
equation as an exact equation follows after this solution. Since the constant terms are nonzero, we seek a transformation of the form

\[ x = u + h(u = x - h) \]
\[ y = v + k(v = y - k) \]

where \( h, k \) satisfy the linear equations

\[ 2h + k - 1 = 0 \]
\[ h - k - 2 = 0. \]

Adding the two equations yields \( 3h - 3 = 0 \) so \( h = 1 \). Then \( k = -1 \). Substitution yields the new linear equation (note \( du = dx, dv = dy \))

\[ (2u + v)du + (u - v)dv = 0. \]

or,

\[ \frac{dv}{du} = \frac{2u + v}{v - u}, \quad \frac{dv}{du} = \frac{2 + v/u}{v/u - 1}, \]

the final equation being homogeneous. Setting \( z = v/u \), we get \( zu = v \) and consequently \( z + u \frac{dz}{du} = \frac{dv}{du} \). After substitution we obtain the equation

\[ z + u \frac{dz}{du} = \frac{2 + z}{z - 1}, \quad \frac{dz}{du} = \frac{2 + z}{z - 1} - \frac{z(z - 1)}{z - 1}, \]
\[ \frac{dz}{du} = \frac{2 + 2z - z^2}{z - 1}, \quad \int \frac{(z - 1)}{2 + 2z - z^2}dz = \int \frac{du}{u}. \]

Using the substitution method \( (w = 2 + 2z - z^2) \), we get

\[ -\frac{1}{2} \ln |2 + 2z - z^2| = \ln |u| + C \]
\[ \ln |2 + 2z - z^2| = -2 \ln |u| + C. \]

Taking the exponential and absorbing the sign of the absolute value into \( C \) yields,

\[ 2 + 2z - z^2 = Cu^{-2}. \]

Finally, we back substitute to obtain

\[ 2 + 2y + 1 \]
\[ x - 1 \]
\[ - \left( \frac{y + 1}{x - 1} \right)^2 = C(x - 1)^{-2}. \]

**Exact solution:** Set \( M = 2x + y - 1 \), \( N = x - y - 2 \). Clearly \( \frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x} \), so the equation is exact. Then we seek \( F(x, y) \) with
\[ \frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N. \]

\[ F = \int M \, dx + g(y) \]
\[ = \int (2x + y - 1) \, dx + g(y) \]
\[ = x^2 + xy - x + g(y). \]

Taking the derivative with respect to \( y \) of both sides and using \( \frac{\partial F}{\partial y} = N \) we obtain

\[ x - y - 2 = x + g'(y) \]
\[ g'(y) = -y - 2 \]
\[ g(y) = -\frac{y^2}{2} - 2y. \]

Consequently,

\[ F(x, y) = x^2 + xy - x - \frac{y^2}{2} - 2y, \]

and the general solution is given by

\[ x^2 + xy - x - \frac{y^2}{2} - 2y = C. \]

Basic algebra demonstrates the equality of the solution obtained here with the solution obtained above.

(4) [25] Solve the IVP

\[ y'' + y = \cos(t), \quad y(0) = 1, \quad y'(0) = 1. \]

Since the characteristic polynomial \( r^2 + 1 \) has roots \( \pm i \), the general solution of the homogeneous equation is

\[ y_h = c_1 \sin(t) + c_2 \cos(t). \]

The method of undetermined coefficients gives us that a particular solution \( y_p \) has the form

\[ y_p(t) = A_0 t \cos(t) + B_0 t \sin(t). \]

Note the additional factor of \( t \) which appears since \( \cos(t) \) and \( \sin(t) \) are solutions of the homogeneous equation. Taking derivatives, we obtain

\[ y'_p(t) = A_0 (-t \sin(t) + \cos(t)) + B_0 (t \cos(t) + \sin(t)) \]
\[ y''_p(t) = \sin(t) (-t A_0 + B_0) + \cos(t) (A_0 + t B_0), \]
and

\[ y_p''(t) = \sin(t)(-A_0) + \cos(t)(-tA_0 + B_0) + \]
\[ \cos(t)(B_0) - \sin(t)(A_0 + tB_0) \]
\[ y_p''(t) = \sin(t)(-2A_0 - tB_0) + \cos(t)(-tA_0 + 2B_0) \]

When plugged back into the original ODE, we obtain

\[-2A_0 \sin(t) + 2B_0 \cos(t) = \cos(t).\]

Then clearly \( A_0 = 0, B_0 = \frac{1}{2} \) and

\[ y_p(t) = \frac{1}{2} t \sin(t). \]

It follows that the general solution to the ODE is given by

\[ y = \frac{1}{2} t \sin(t) + c_1 \sin(t) + c_2 \cos(t) \]

which has derivative

\[ y' = \frac{1}{2} (t \cos(t) + \sin(t)) + c_1 \cos(t) - c_2 \sin(t) \]

Plugging in \( y(0) = 1 \) immediately yields \( 1 = c_2 \). Then plugging in \( y'(0) = 1 \) gives \( 1 = c_1 \). Then the solution which satisfies the IVP is

\[ y = \frac{1}{2} t \sin(t) + \sin(t) + \cos(t). \]

(5) [20] Use variation of parameters to find the general solution to

\[ y'' - 2y' + y = \frac{e^t}{t^2 + 1} \]

The associated homogeneous equation has \( y_1(t) = e^t, y_2(t) = te^t \) as linearly independent solutions. The solution given by variation of parameters has the form

\[ y_p = v_1y_1 + v_2y_2 \]

where \( v_1, v_2 \) satisfy the system

\[ v_1'e^t + v_2'te^t = 0 \]
\[ v_1'e^t + v_2'e^t(t + 1) = \frac{e^t}{t^2 + 1}. \]

Subtracting the second equation from the first yields

\[ v_2'(te^t - (t + 1)e^t) = \frac{-e^t}{t^2 + 1} \]
\[ v_2' = \frac{1}{t^2 + 1}. \]
Substituting back into the first equation then gives \( v_1' = -\frac{t}{t^2 + 1} \).

Integrating yields
\[
v_1 = -\frac{1}{2} \ln |t^2 + 1| = -\frac{1}{2} \ln (t^2 + 1)
\]
\[
v_2 = \arctan(t).
\]

(These could be found by memorizing the formulas too). A solution \( y_p \) is then given by
\[
y_p(t) = -\frac{1}{2} \ln (t^2 + 1) e^t + \arctan(t)te^t.
\]

Consequently the general solution is given by
\[
y = -\frac{1}{2} \ln (t^2 + 1) e^t + \arctan(t)te^t + c_1 e^t + c_2 te^t.
\]

(6) **Bonus +10** Use Euler’s formula
\[
e^{i\theta} = \sin(\theta) + i \cos(\theta)
\]
to prove that
\[
\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}.
\]

In turn, assuming complex exponents work like real exponents use this to prove the double angle formula
\[
\cos(2\theta) = 2 \cos^2(\theta) - 1.
\]

Most of the trig identities that you had to memorize are established in this way!

Since \( \cos(-\theta) = \cos(\theta) \) and \( \sin(-\theta) = -\sin(\theta) \), the first identity follows. Now
\[
\cos(2\theta) = \frac{e^{2i\theta} + e^{-2i\theta}}{2}
\]
\[
= \frac{e^{2i\theta} + e^{-2i\theta} + 2e^{-i}e^i - 2e^{-i}e^i}{2}
\]
\[
= \frac{e^{2i\theta} + 2e^{-i}e^i + e^{-2i\theta}}{2} - \frac{2e^{-i}e^i}{2}
\]
\[
= \frac{(e^{i\theta} + e^{-i\theta})^2}{2} - 1
\]
\[
= 2 \cos^2(\theta) - 1
\]