(1) Find the general solution to the following ODEs
(a) \(t^2y'' + 7ty' - 7y = 0\) for \(t > 0\)
The characteristic polynomial \(r^2 + 6r - 7 = 0\) has roots \(r = -7, 1\).
Then the general solution is
\[y = c_1 t^{-7} + c_2 t.\]

(b) \(y^{(7)} - y^{(6)} + 8y^{(5)} - 8y^{(4)} + 16y''' - 16y'' = 0\) assuming
\(r^7 - r^6 + 8r^5 - 8r^4 + 16r^3 - 16r^2 = r^2(r - 1)(r^2 + 4)^2\)
The characteristic polynomial is given above. The double root \(r = 0\) gives rise to solutions \(e^{0t} = 1\) and \(te^{0t} = t\). The single root \(r = 1\) gives \(e^{t}\) as a solution, and the irreducible repeated quadratic term gives the \(\sin(2t), \cos(2t), t\sin(2t), t\cos(2t)\) as solutions. Then general solution then is
\[y = c_1 + c_2 t + c_3 e^t + c_4 \sin(2t) + c_5 \cos(2t) + c_6 t \sin(2t) + c_7 t \cos(2t).\]

(c) \(t^2y'' + ty' + y = 0\) for \(t > 0\)
The characteristic polynomial \(r^2 + 1\) has roots \(r = \pm i\) and so the general solution is
\[y = c_1 \cos(\ln(t)) + c_2 \sin(\ln(t)).\]

(2) Use reduction of order to find a second linearly independent solution \(y_2\) to the ODE
\((t - 1)y'' - ty' + y = 0\)
for \(t > 1\) given that \(y_1(t) = e^t\) is a solution. Use the Wronskian of \(y_1, y_2\) to verify that they are linearly independent.
In standard form the differential equation is
\[y'' - \frac{t}{t - 1} y' + \frac{y}{t - 1} = 0.\]
The reduction of order formula gives
\[y_2 = e^t \int \frac{e^{\int \frac{t}{t - 1} dt}}{e^{2t}} dt\]
\[y_2 = e^t \int \frac{e^{t + \ln |t - 1|}}{e^{2t}} dt\]
Since \(t > 1\), we have \(|t - 1| = t - 1\) and so
\[y_2 = e^t \int \frac{e^t(t - 1)}{e^{2t}} dt = e^t \int e^{-t}(t - 1) dt\]
After doing integration by parts we obtain
\[ y_2 = e^t(te^{-t}) = t. \]
\[ W[e^t, t] = e^t - te^t = e^t(1 - t) \]
which is nonzero for \( t > 1 \), and so the functions are linearly independent on this interval.

(3) A mass of 2kg is attached to a spring with stiffness 50N/m. The mass is initially placed 1m to the left of the spring’s equilibrium point and given an initial velocity of 1m/s to the left. Assume that the damping force is negligible. Find the equation of motion for the mass as well as the amplitude, period, and frequency. How long after release does the mass pass through the equilibrium point of the spring?

The ODE governing this system is
\[ my'' + ky = 0. \]
Initial conditions are given by \( y(0) = -1, y'(0) = -1 \). With \( m = 2, k = 50 \) this becomes
\[ 2y'' + 50y = 0. \]
The characteristic equation has roots \( r = \pm 5 \) and so the general solution is given by
\[ y = c_1 \sin(5t) + c_2 \cos(5t). \]
Then its derivative is given by
\[ y' = 5c_1 \cos(5t) - 5c_2 \sin(5t). \]
Plugging in the initial conditions yields \(-1 = c_2, -1/5 = c_1\). So the equation of motion is given by
\[ y = (-1/5) \sin(5t) - \cos(5t). \]
We can immediately say that the period is \( \frac{2\pi}{5} \) and so the frequency is \( \frac{5}{2\pi} \). To find the amplitude, we rewrite \( y \) as
\[ y = A \sin(5t + \phi) \]
where \( A = \sqrt{c_1^2 + c_2^2} \) and \( \tan(\phi) = \frac{c_2}{c_1} \) (where \( \phi \) must be selected from the appropriate quadrant). Here we are only asked to find the amplitude \( A \) so
\[ A = \sqrt{1 + (1/25)} = \sqrt{26}/25. \]
Finding \( \phi \) is impossible by hand since I cannot find \( \arctan(5) \). To find when the mass passes through the equilibrium point we need to solve \( y(t) = 0 \) and find the smallest positive \( t \) which satisfies this.
\[ 0 = (-1/5) \sin(5t) - \cos(5t) \]
\[ -5 = \tan(5t) \]
\[ \arctan(-5) + k\pi = 5t \]
Here \( k \) is an arbitrary integer
\[ \frac{\arctan(-5) + k\pi}{5} = t \]
We have $-\pi/2 < \arctan(-5) < 0$ so that $k = 0$ gives a negative $t$. But then $(k = 1) \arctan(-5) + \pi > 0$. Hence $t = \frac{\arctan(-5) + \pi}{5}$.

(4) Suppose that $t, t^2$, and $t^3$ are solutions to an ODE $y'' + p(t)y' + q(t)y = g(t)$. Find a solution to that ODE which satisfies $y(2) = 2, y'(2) = 5$.

By the superposition principle $y_1 = t^2 - t, y_2 = t^3 - t$ are solutions to the associated homogeneous equation. That these are linearly independent is easily verified. Then the general solution is given by

$$y = c_1(t^2 - t) + c_2(t^3 - t)$$

with derivative

$$y' = 1 + c_1(2t - 1) + c_2(3t^2 - 1).$$

Plugging in the initial conditions yields the equations

$$0 = 2c_1 + 6c_2$$
$$4 = 3c_1 + 11c_2$$

Which is solved to yield $c_1 = -6, c_2 = 2$. Then the solution to the IVP is

$$y = t - 6(t^2 - t) + 2(t^3 - t)$$

(5) Find the general solution to the system

$$x' = x - y$$
$$y' = y - 4x$$

In differential operator form this is the system

$$(D - 1)[x] + [y] = 0$$
$$4[x] + (D - 1)[y] = 0$$

Multiplying the first by $-4$ and applying $D - 1$ to the second yields

$$-4(D - 1)[x] - 4[y] = 0$$
$$4(D - 1)[x] + (D - 1)^2[y] = 0$$

Adding these equations gives the uncoupled equation

$$(D^2 - 2D - 3)[y] = 0.$$  

Equivalently, this is

$$(D - 3)(D + 1)[y] = 0.$$  

Since the characteristic polynomial has roots $3, -1$ we have the general form

$$y = c_1e^{3t} + c_2e^{-t}.$$  

Observing from the original problem that $x = (y - y')/4$ and that from the above we have

$$y' = 3c_1e^{3t} - c_2e^{-t},$$

we can obtain that

$$x = (-1/2)c_1e^{3t} + (1/2)c_2e^{-t}.$$
The obtained equations for \( x, y \) give the general solution.

(6) Does the Laplace transform exist for the following functions? Justify your answers. Recall that the domain of the functions should be taken to be \([0, \infty)\).

(a) \( e^t \)  
(b) \( \frac{1}{t^2+1} \)  
(c) \( \frac{1}{t-1} \)  
(d) \( e^{\frac{t^2}{t^2+1}} \)  
(e) \( f(t) = \begin{cases} 
0 & \text{if } t < 5 \\
e^{t^2} & \text{if } 5 \leq t < 10 \\
1/t & \text{if } t \geq 10
\end{cases} \)

(a) Yes, for \( s > 1 \) since it is continuous and trivially of exponential order 1.
(b) Yes, for \( s > 0 \). This function is clearly continuous. Since 1 is of exponential order \( s \) for all \( s > 0 \) and the function is less than or equal to 1, it to is of exponential order \( s \) for all \( s > 0 \).
(c) No, the function has a vertical asymptote at \( t = 1 \).
(d) Yes, for \( s > 1 \). \( \lim_{t \to \infty} \frac{t}{t^2+1} = 1 \). Consequently, for any \( s > 1 \) and \( t \) sufficiently large, \( \frac{t}{t^2+1} \leq s \) and so \( e^{\frac{t^2}{t^2+1}} \leq e^{ts} \), showing that the function is of exponential order \( s \).
(e) Yes, for \( s > 0 \). The function is piecewise continuous (the asymptote of \( 1/t \) is not an issue since the function is only defined as \( 1/t \) for \( t \geq 10 \). Since 1 is of exponential order \( s \) for all \( s > 0 \) and \( 1/t \leq 1 \) for all \( t \geq 10 \), this function is of exponential order \( s \) for all \( s > 0 \).