(1) Find a differential equation of the form $\frac{dy}{dx} = G(y)$ so that $y = \tan(x)$ is a solution.

We have that $\frac{d}{dx}\tan(x) = \sec^2(x) = \tan^2(x) + 1 = y^2 + 1$. So $\tan(x)$ is a solution to the DE

$$\frac{dy}{dx} = y^2 + 1.$$ 

(2) Apply the transformation $u = xy$ to the differential equation

$$\frac{dy}{dx} = \frac{e^{xy} - xy}{x^2}.$$ 

Use this to solve the DE.

If $u = xy$ then

$$\frac{du}{dx} = x\frac{dy}{dx} + y$$

$$\frac{du}{dx} - y = x\frac{dy}{dx}$$

$$\frac{du}{dx} - u = x\frac{dy}{dx}$$

$$\frac{du}{dx} - u = x^2\frac{dy}{dx}.$$ 

We can rewrite the DE as

$$x^2\frac{dy}{dx} = e^{xy} - xy.$$ 

Applying the substitution we have

$$x\frac{du}{dx} - u = e^u - u$$

$$x\frac{du}{dx} = e^u.$$ 

This equation is separable, so we can solve it by separating and integrating.

$$e^{-u}du = \frac{1}{x}dx$$

$$-e^{-u} = \ln|x| + C$$

Backsubstitution gives

$$-e^{-xy} = \ln|x| + C$$
as a family of solutions.

(3) Solve the IVP

\[
\frac{e^x}{y^2 + 1} \, dy - x \, dx = 0 \quad y(0) = 0.
\]

This may appear to be solved by the method for exact equations, but it is actually separable.

\[
\frac{e^x}{y^2 + 1} \, dy - x \, dx = 0
\]

\[
\frac{1}{y^2 + 1} \, dy = e^{-x} \, x \, dx.
\]

Integrating both sides gives

\[
\tan^{-1}(y) = -xe^{-x} - e^{-x} + C.
\]

Using the initial condition \( y(0) = 0 \) we have \( C = 0 \) so the solution is given implicitly by

\[
\tan^{-1}(y) = -xe^{-x} - e^{-x} + 1
\]

and explicitly by

\[
y = \tan(-xe^{-x} - e^{-x} + 1).
\]

(4) Find the most general family of solutions to the differential equation

\[
x \frac{dy}{dx} - (1 + x)y = xy^2
\]

This is a Bernoulli equation. After dividing by \( x \) and \( y^2 \) we have

\[
y^{-2} \frac{dy}{dx} - \frac{1 + x}{x} y^{-1} = 1
\]

Letting \( u = y^{-1} \), we have \( \frac{du}{dx} = -y^{-2} \frac{dy}{dx} \). Substitution results in the equation

\[
- \frac{du}{dx} - \frac{1 + x}{x} u = 1
\]

\[
\frac{du}{dx} + \frac{1 + x}{x} u = -1
\]

The resulting equation is linear and in standard form. We choose integrating factor \( \mu = e^{\int \frac{1+x}{x} \, dx} = e^{x+\ln(x)} = xe^x \). After multiplication
by $\mu$ we have
\[
xe^x \frac{du}{dx} + e^x (1 + x) u = -xe^x \\
\frac{d}{dx} (xe^x u) = -xe^x \\
x e^x u = - \int xe^x \, dx = -xe^x + e^x + C \\
u = -1 + \frac{1}{x} + \frac{C}{xe^x}
\]
Backsubstitution for $y$ yields
\[
\frac{1}{y} = -1 + \frac{1}{x} + \frac{C}{xe^x}
\]
as a family of implicit solutions.

(5) Find an integrating factor of the form $x^n y^m$ to the ODE
\[
(12 + 5xy)dx + (6xy^{-1} + 3x^2)dy = 0.
\]
Use this to find a family of solutions to the ODE.

Set $\mu(x,y) = x^n y^m$ for unknown $n, m$. After multiplying through my $\mu$ we have
\[
(12x^n y^m + 5x^{n+1} y^{m+1})dx + (6x^{n+1} y^{m-1} + 3x^{n+2} y^m)dy = 0.
\]
The equation is exact if and only if
\[
\frac{\partial}{\partial y}(12x^n y^m + 5x^{n+1} y^{m+1}) = \frac{\partial}{\partial x}(6x^{n+1} y^{m-1} + 3x^{n+2} y^m)
\]
\[
12mx^n y^{m-1} + 5(m + 1)x^{n+1} y^m = 6(n + 1)x^n y^{m-1} + 3(n + 2)x^{n+1} y^m
\]
Equating coefficients of $x^n y^{m-1}$ and $x^{n+1} y^m$ in the above equation gives the following system of linear equations:
\[
12m = 6(n + 1) \\
5(m + 1) = 3(n + 2)
\]
Solving this gives $m = 2, n = 3$ so $\mu(x,y) = x^3 y^2$. After multiplying through my $\mu$ we have the equation
\[
(12x^3 y^2 + 5x^4 y^3)dx + (6x^4 y + 3x^5 y^2)dy = 0
\]
This equation is exact, so we try to find $F(x,y)$ so that $F(x,y) = C$ gives a family of implicit solutions.
\[
F(x,y) = \int (12x^3 y^2 + 5x^4 y^3) \, dx \\
F(x,y) = 3x^4 y^2 + x^5 y^3 + g(y).
\]
To determine $g(y)$ we differentiate with respect to $y$.

$$\frac{\partial F}{\partial y} = 6x^4y + 3x^5y^2 + g'(y).$$

Since the DE is exact we can plug in for $\frac{\partial F}{\partial y}$ and obtain

$$6x^4y + 3x^5y^2 = 6x^4y + 3x^5y^2 + g'(y)$$

$$0 = g'(y)$$

$$0 = g(y)$$

Hence $F(x, y) = 3x^4y^2 + x^5y^3$ and a family of solutions is given by $3x^4y^2 + x^5y^3 = C$. 