MAP 2302, Exam I, Fall 2015

Name:____________________

Student signature:____________________

Turn in all relevant work with final answers circled on separate sheets. Full work is required for full credit.

(1) Find a differential equation of the form $\frac{dy}{dx} = G(y)$ so that $y = \tan(x)$ is a solution.

We have that $\frac{d}{dx}\tan(x) = \sec^2(x) = \tan^2(x) + 1 = y^2 + 1$. So $\tan(x)$ is a solution to the DE

$$\frac{dy}{dx} = y^2 + 1.$$

(2) Apply the transformation $u = xy$ to the differential equation

$$\frac{dy}{dx} = \frac{e^{xy} - xy}{x^2}.$$

Use this to solve the DE.

If $u = xy$ then

$$\begin{aligned} \frac{du}{dx} &= x\frac{dy}{dx} + y \\ \frac{du}{dx} - y &= x\frac{dy}{dx} \\ \frac{du}{dx} - \frac{u}{x} &= x\frac{dy}{dx} \\ x\frac{du}{dx} - u &= x^2\frac{dy}{dx}. \end{aligned}$$

We can rewrite the DE as

$$x^2\frac{dy}{dx} = e^{xy} - xy$$

Applying the substitution we have

$$\begin{aligned} x\frac{du}{dx} - u &= e^u - u \\ x\frac{du}{dx} &= e^u. \end{aligned}$$

This equation is separable, so we can solve it by separating and integrating.

$$\begin{aligned} e^{-u}du &= \frac{1}{x}dx \\ -e^{-u} &= \ln|x| + C \end{aligned}$$

Backsubstitution gives

$$-e^{-xy} = \ln|x| + C$$

as a family of solutions.

(3) Solve the IVP

$$\frac{e^x}{y^2+1}dy - xdx = 0 \quad y(0) = 0.$$

This may appear to be solved by the method for exact equations, but it is actually separable.

$$\frac{e^x}{y^2+1}dy - xdx = 0$$
$$\frac{1}{y^2+1}dy = e^{-x}xdx.$$

Integrating both sides gives

$$\tan^{-1}(y) = -xe^{-x} - e^{-x} + C.$$

Using the initial condition $y(0) = 0$ we have $C = 1$ so the solution is given implicitly by

$$\tan^{-1}(y) = -xe^{-x} - e^{-x} + 1$$

and explicitly by

$$y = \tan(-xe^{-x} - e^{-x} + 1).$$

(4) Find the most general family of solutions to the differential equation

$$x\frac{dy}{dx} - (1+x)y = xy^2$$

This is a Bernoulli equation. After dividing by x and y^2 we have

$$y^{-2}\frac{dy}{dx} - \frac{1+x}{x}y^{-1} = 1$$

Letting $u = y^{-1}$, we have $\frac{du}{dx} = -y^{-2}\frac{dy}{dx}$. Substitution results in the equation

$$-\frac{du}{dx} - \frac{1+x}{x}u = 1$$
$$\frac{du}{dx} + \frac{1+x}{x}u = -1$$

The resulting equation is linear and in standard form. We choose integrating factor $\mu = e^{\int \frac{1+x}{x}dx} = e^{x+\ln(x)} = xe^x$. After multiplication by μ we have

$$\begin{aligned} xe^x\frac{du}{dx} + e^x(1+x)u &= -xe^x \\ \frac{d}{dx}\left(xe^x u\right) &= -xe^x \\ xe^x u &= -\int xe^x\, dx = -xe^x + e^x + C \\ u &= -1 + \frac{1}{x} + \frac{C}{xe^x} \end{aligned}$$

Backsubstitution for y yields

$$\frac{1}{y} = -1 + \frac{1}{x} + \frac{C}{xe^x}$$

as a family of implicit solutions.

(5) Find an integrating factor of the form $x^n y^m$ to the ODE

$$(12 + 5xy)dx + (6xy^{-1} + 3x^2)dy = 0.$$

Use this to find a family of solutions to the ODE.

Set $\mu(x, y) = x^n y^m$ for unknown n, m. After multiplying through my μ we have

$$(12x^n y^m + 5x^{n+1}y^{m+1})dx + (6x^{n+1}y^{m-1} + 3x^{n+2}y^m)dy = 0.$$

The equation is exact if and only if

$$\begin{aligned} \frac{\partial}{\partial y}(12x^n y^m + 5x^{n+1}y^{m+1}) &= \frac{\partial}{\partial x}(6x^{n+1}y^{m-1} + 3x^{n+2}y^m) \\ 12mx^n y^{m-1} + 5(m+1)x^{n+1}y^m &= 6(n+1)x^n y^{m-1} + 3(n+2)x^{n+1}y^m \end{aligned}$$

Equating coefficients of $x^n y^{m-1}$ and $x^{n+1}y^m$ in the above equation gives the following system of linear equations:

$$\begin{aligned} 12m &= 6(n+1) \\ 5(m+1) &= 3(n+2) \end{aligned}$$

Solving this gives $m = 2, n = 3$ so $\mu(x, y) = x^3y^2$. After multiplying through my μ we have the equation

$$(12x^3y^2 + 5x^4y^3)dx + (6x^4y + 3x^5y^2)dy = 0$$

This equation is exact, so we try to find $F(x, y)$ so that $F(x, y) = C$ gives a family of implicit solutions.

$$\begin{aligned} F(x, y) &= \int (12x^3y^2 + 5x^4y^3)\, dx \\ F(x, y) &= 3x^4y^2 + x^5y^3 + g(y). \end{aligned}$$

To determine $g(y)$ we differentiate with respect to y.

$$\frac{\partial F}{\partial y} = 6x^4y + 3x^5y^2 + g'(y).$$

Since the DE is exact we can plug in for $\frac{\partial F}{\partial y}$ and obtain

$$\begin{aligned} 6x^4y + 3x^5y^2 &= 6x^4y + 3x^5y^2 + g'(y) \\ 0 &= g'(y) \\ 0 &= g(y) \end{aligned}$$

Hence $F(x, y) = 3x^4y^2 + x^5y^3$ and a family of solutions is given by $3x^4y^2 + x^5y^3 = C$.