(1) [25] The graph of a function $f$ is shown below.

(a) [10] Sketch a direction field for the differential equation $\frac{dy}{dx} = f(y)$. Your sketch must show points with $0 \leq x \leq 3$ and $-2 \leq y \leq 2$ and show directions at each integer point. Slopes are computed by plugging the $y$-coordinate of the point into $f$. These are shown on the graph below.

(b) [10] Estimate the constant solutions of the above differential equation. Draw them on your direction field. Constant solutions occur when $\frac{dy}{dx} = 0$ or equivalently $f(y) = 0$. This happens when $y \approx \pm 1.5$. These are drawn in red on the graph below.

(c) [5] Use your direction field to sketch two solution curves, one satisfying $y(0) = 2$ and the other satisfying $y(0) = 0$. With the direction field drawn, we need only follow the arrows and avoid crossing already computed solutions. These are shown in blue on the below graph.
(2) [25] Find the most general family of solutions to the equation [15]

\[ x \frac{dy}{dx} - 3y = x^4 \]

What are the singular points of the equation [5]? Does the equation have a unique solution satisfying \( y(0) = 0 \) [5]? Justify your answer.

First, we have to put this linear equation in standard form

\[ \frac{dy}{dx} - \frac{3}{x} y = x^3 \]

Since we have a singular point at \( x = 0 \), we can only solve on either \((0, \infty)\) or \((-\infty, 0)\). In either case, an integrating factor is given by

\[ \mu(x) = e^{\int \left(-\frac{3}{x}\right) \, dx} = e^{\ln|x|^{-3} + C} = C|x|^{-3} = Cx^{-3}. \]

We may choose \( C = 1 \) so that \( \mu(x) = x^{-3} \) regardless of whether \( x > 0 \) or \( x < 0 \). Then we can multiply through by the integrating factor to obtain.

\[ x^{-3} \frac{dy}{dx} - 3x^{-4} y = 1 \]

\[ \frac{d}{dx} \left( x^{-3} y \right) = 1 \]

\[ x^{-3} y = x + C \]

\[ y = x^4 + Cx^3 \]

Since we have solved independently for \( x < 0 \) and \( x > 0 \) (albeit with the same technique) we can choose different constants for \( x < 0 \) and \( x > 0 \) so the most general family of solutions is given by the piecewise defined family

\[ y = \begin{cases} 
  x^4 + Cx^3 & \text{if } x > 0 \\
  x^4 + Dx^3 & \text{if } x \leq 0
\end{cases} \]

Observe that this is well-defined at 0 as both sides would have \( y(0) = 0 \) regardless of choice of \( C, D \).
On the exam, I gave full credit for saying \( y = x^4 + C x^3 \) was the family of solutions, \( x = 0 \) was a singular point, and the solution was not unique because \( y(0) = 0 \) regardless of choice of \( C \). Note that the full solution is a bit more complicated.

(3) [25] Find a solution to the IVP

\[
\frac{dy}{dx} = y + \sqrt{x^2 - y^2} ; y(1) = 0
\]

You may use the fact that \( \int \frac{1}{\sqrt{1 - x^2}} \, dx = \sin^{-1}(x) \) without work.

After exhausting all other methods, we realize that the equation is homogeneous and use the substitution \( y = ux \) so that \( \frac{dy}{dx} = u + x \frac{du}{dx} \). We will rewrite the equation by dividing by \( x \). Once again, by doing this we will have to choose to solve on \( x > 0 \) or \( x < 0 \). Since our initial value is given at \( x = 1 \) we will choose \( x > 0 \). Division yields

\[
\frac{dy}{dx} = \frac{y}{x} + \frac{\sqrt{x^2 - y^2}}{x}.
\]

After substitution we have

\[
\frac{du}{dx} = \frac{u + \sqrt{x^2 - x^2 u^2}}{x} = \frac{2}{x} \sqrt{1 - u^2}.
\]

Since \( x > 0 \), \( |x| = x \) and \( |x|/x = 1 \). After separating we have

\[
\frac{1}{\sqrt{1 - u^2}} \, du = \frac{1}{x} \, dx
\]

\[
\sin^{-1}(u) = \ln(x) + C.
\]

In the last step, we have again used our assumption that \( x > 0 \). Then changing back from \( u \) to \( y \) we have

\[
\sin^{-1}(y/x) = \ln(x) + C.
\]

With the initial condition \( y(1) = 0 \) we find \( C = 0 \) so the solution is

\[
\sin^{-1}(y/x) = \ln(x).
\]

(4) [25] Find the most general family of solutions to the equation

\[
(3x^2 + y) \, dx + (x^2 y - x) \, dy = 0.
\]

We can solve this using integrating factors for exact equations. Letting \( M = 3x^2 + y \) and \( N = x^2 y - x \) we have that \( M_y = 1 \) and \( N_x = 2xy - 1 \). Then

\[
\frac{M_y - N_x}{N} = \frac{2 - 2xy}{x^2 y - x} = \frac{-2(xy - 1)}{x (xy - 1)} = \frac{-2}{x}.
\]
Consequently, we can find our integrating factor as
\[ \mu(x) = e^{\int \frac{-2}{x} \, dx} = x^{-2}. \]

After multiplying through by \( \mu \) we have the equation
\[ (3 + x^{-2}y) \, dx + (y - x^{-1}) \, dy = 0. \]

This equation is exact. Then we look for solutions of the form
\[ F(x, y) = C. \]
We know that \( F_x = 3 + x^{-2}y \) and \( F_y = y - x^{-1} \).
Integrating \( F_x \) we get
\[ F = 3x - x^{-1}y + g(y). \]

Taking the \( y \) partial derivative we have
\[ F_y = x^{-1}y + g'(y) \]

Since \( F_y = y - x^{-1} \)
\[ y - x^{-1} = x^{-1}y + g'(y) \]
\[ g'(y) = y \]
\[ g(y) = \frac{y^2}{2} \]

Then \( F(x, y) = 3x - x^{-1}y + \frac{y^2}{2} \) and solutions are given implicitly by
\[ 3x - x^{-1}y + \frac{y^2}{2} = C. \]