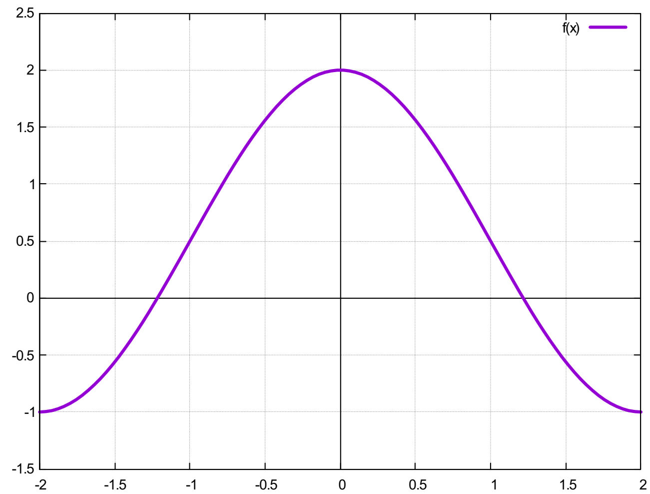
MAP 2302, Exam I, Fall 2015

Name:________________________

Student signature:________________________

Turn in all relevant work with final answers circled on separate sheets. Full work is required for full credit.

(1) [25] The graph of a function f is shown below.



(a) [10] Sketch a direction field for the differential equation $\frac{dy}{dx} = f(y)$. Your sketch must show points with $0 \leq x \leq 3$ and $-2 \leq y \leq 2$ and show directions at each integer point.
Slopes are computed by plugging the y-coordinate of the point into f. These are shown on the graph below.

(b) [10] Estimate the constant solutions of the above differential equation. Draw them on your direction field.
Constant solutions occur when $\frac{dy}{dx} = 0$ or equivalently $f(y) = 0$. This happens when $y \approx \pm 1.5$. These are drawn in red on the graph below.

(c) [5] Use your direction field to sketch two solution curves, one satisfying $y(0) = 2$ and the other satisfying $y(0) = 0$.
With the direction field drawn, we need only follow the arrows and avoid crossing already computed solutions. these are shown in blue on the below graph.

(2) [25] Find the most general family of solutions to the equation [15]

$$x\frac{dy}{dx} - 3y = x^4$$

What are the singular points of the equation [5]? Does the equation have a unique solution satisfying $y(0) = 0$ [5]? Justify your answer.

First, we have to put this linear equation in standard form

$$\frac{dy}{dx} - \frac{3}{x}y = x^3$$

Since we have a singular point at $x = 0$, we can only solve on either $(0, \infty)$ or $(-\infty, 0)$. In either case, an integrating factor is given by $\mu(x) = e^{\int (-3/x)\,dx} = e^{ln|x|^{-3}+C} = C|x|^{-3} = Cx^{-3}$. We may choose $C = 1$ so that $\mu(x) = x^{-3}$ regardless of whether $x > 0$ or $x < 0$. Then we can multiply through by the integrating factor to obtain.

$$\begin{aligned} x^{-3}\frac{dy}{dx} - 3x^{-4}y &= 1 \\ \frac{d}{dx}\left(x^{-3}y\right) &= 1 \\ x^{-}3y &= x + C \\ y &= x^4 + Cx^3 \end{aligned}$$

Since we have solved independently for $x < 0$ and $x > 0$ (albeit with the same technique) we can choose different constants for $x < 0$ and $x > 0$ so the most general family of solutions is given by the piecewise defined family

$$y = \begin{cases} x^4 + Cx^3 & \text{if } x > 0 \\ x^4 + Dx^3 & \text{if } x \leq 0 \end{cases}.$$

Observe that this is well-defined at 0 as both sides would have $y(0) = 0$ regardless of choice of C, D.

On the exam, I gave full credit for saying $y = x^4 + Cx^3$ was the family of solutions, $x = 0$ was a singular point, and the solution was not unique because $y(0) = 0$ regardless of choice of C. Note that the full solution is a bit more complicated.

(3) [25] Find a solution to the IVP

$$x\frac{dy}{dx} = y + \sqrt{x^2 - y^2} \;\; ; y(1) = 0$$

You may use the fact that $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x)$ without work.

After exhausting all other methods, we realize that the equation is homogeneous and use the substitution $y = ux$ so that $\frac{dy}{dx} = u + x\frac{du}{dX}$. We will rewrite the equation by dividing by x. Once again, by doing this we will have to choose to solve on $x > 0$ or $x < 0$. Since our initial value is given at $x = 1$ we will choose $x > 0$. Division yields

$$\frac{dy}{dx} = \frac{y}{x} + \frac{\sqrt{x^2 - y^2}}{x}.$$

After substitution we have

$$\begin{aligned} u + x\frac{du}{dx} &= u + \frac{\sqrt{x^2 - x^2u^2}}{x} \\ x\frac{du}{dx} &= \frac{|x|\sqrt{1-u^2}}{x}. \end{aligned}$$

Since $x > 0$, $|x| = x$ and $|x|/x = 1$. After separating we have.

$$\frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{x}\,dx$$
$$\sin^{-1}(u) = \ln(x) + C.$$

In the last step, we have again used our assumption that $x > 0$. Then changing back from u to y we have

$$\sin^{-1}(y/x) = \ln(x) + C.$$

With the initial condition $y(1) = 0$ we find $C = 0$ so the solution is

$$\sin^{-1}(y/x) = \ln(x).$$

(4) [25] Find the most general family of solutions to the equation

$$(3x^2 + y)\,dx + (x^2y - x), dy = 0.$$

We can solve this using integrating factors for exact equations. Letting $M = 3x^2 + y$ and $N = x^2y - x$ we have that $M_y = 1$ and $N_x = 2xy - 1$. Then

$$\frac{M_y - N_x}{N} = \frac{2 - 2xy}{x^2y - x} = \frac{-2(xy-1)}{x(xy-1)} = \frac{-2}{x}.$$

Consequently, we can find our integrating factor as

$$\mu(x) = e^{\int(-2/x)\,dx} = x^{-2}.$$

After multiplying through my μ we have the equation

$$(3 + x^{-2}y)\,dx + (y - x^{-1})\,dy = 0.$$

This equation is exact. Then we look for solutions of the form $F(x,y) = C$. We know that $F_x = 3 + x^{-2}y$ and $F_y = y - x^{-1}$. Integrating F_x we get

$$F = 3x - x^{-1}y + g(y).$$

Taking the y partial derivative we have

$$F_y = x^{-1}y + g'(y)$$

Since $F_y = y - x^{-1}$

$$y - x^{-1} = x^{-1}y + g'(y)$$
$$g'(y) = y$$
$$g(y) = \frac{y^2}{2}$$

Then $F(x,y) = 3x - x^{-1}y + \frac{y^2}{2}$ and solutions are given implicitely by

$$3x - x^{-1}y + \frac{y^2}{2} = C.$$