(1) [25] The graph of a function $f$ is shown below.

(a) [10] Sketch a direction field for the differential equation $\frac{dy}{dx} = f(y)$. Your sketch must show points with $0 \leq x \leq 3$ and $-2 \leq y \leq 2$ and show directions at each integer point. Slopes are computed by plugging the $y$-coordinate of the point into $f$. These are shown on the graph below.

(b) [10] Estimate the constant solutions of the above differential equation. Draw them on your direction field. Constant solutions occur when $\frac{dy}{dx} = 0$ or equivalently $f(y) = 0$. This happens when $y \approx \pm 1.5$. These are drawn in red on the graph below.

(c) [5] Use your direction field to sketch two solution curves, one satisfying $y(0) = 2$ and the other satisfying $y(0) = 0$. With the direction field drawn, we need only follow the arrows and avoid crossing already computed solutions. These are shown in blue on the below graph.

(2) [25] Find the most general family of solutions to the equation [15]

$$x \frac{dy}{dx} - 3y = x^4$$

What are the singular points of the equation [5]? Does the equation have a unique solution satisfying $y(0) = 0$ [5]? Justify your answer.

First, we have to put this linear equation in standard form

$$\frac{dy}{dx} - \frac{3}{x}y = x^3$$
Since we have a singular point at \( x = 0 \), we can only solve on either \((0, \infty)\) or \((-\infty, 0)\). In either case, an integrating factor is given by 
\[
\mu(x) = e^{\int \left(-\frac{3}{x}\right) \, dx} = e^{\ln|x|^{-3} + C} = C|x|^{-3} = Cx^{-3}. \]
We may choose \( C = 1 \) so that \( \mu(x) = x^{-3} \) regardless of whether \( x > 0 \) or \( x < 0 \). Then we can multiply through by the integrating factor to obtain.
\[
x^{-3} \frac{dy}{dx} - 3x^{-4} y = 1
\]
\[
\frac{d}{dx} (x^{-3} y) = 1
\]
\[
x^{-3} y = x + C
\]
\[
y = x^4 + Cx^3
\]
Since we have solved independently for \( x < 0 \) and \( x > 0 \) (albeit with the same technique) we can choose different constants for \( x < 0 \) and \( x > 0 \) so the most general family of solutions is given by the piecewise defined family
\[
y = \begin{cases} 
x^4 + Cx^3 & \text{if } x > 0 \\
x^4 + Dx^3 & \text{if } x \leq 0
\end{cases}
\]
Observe that this is well-defined at 0 as both sides would have \( y(0) = 0 \) regardless of choice of \( C, D \).

On the exam, I gave full credit for saying \( y = x^4 + Cx^3 \) was the family of solutions, \( x = 0 \) was a singular point, and the solution was not unique because \( y(0) = 0 \) regardless of choice of \( C \). Note that the full solution is a bit more complicated.

(3) [25] Find a solution to the IVP
\[
x \frac{dy}{dx} = y + \sqrt{x^2 - y^2} ; y(1) = 0
\]
You may use the fact that \( \int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1}(x) \) without work.

After exhausting all other methods, we realize that the equation is homogeneous and use the substitution \( y = ux \) so that \( \frac{dy}{dx} = u + x \frac{du}{dx} \). We will rewrite the equation by dividing by \( x \). Once again, by doing this we will have to choose to solve on \( x > 0 \) or \( x < 0 \). Since our initial value is given at \( x = 1 \) we will choose \( x > 0 \). Division yields
\[
\frac{dy}{dx} = \frac{y}{x} + \frac{\sqrt{x^2 - y^2}}{x}.
\]
After substitution we have
\[
u + x \frac{du}{dx} = u + \frac{\sqrt{x^2 - x^2u^2}}{x}
\]
\[
x \frac{du}{dx} = |x| \frac{\sqrt{1 - u^2}}{x}.
\]
Since \( x > 0 \), \(|x| = x\) and \(|x|/x = 1\). After separating we have
\[
\frac{1}{\sqrt{1 - u^2}} \, du = \frac{1}{x} \, dx
\]
\[
\sin^{-1}(u) = \ln(x) + C.
\]
In the last step, we have again used our assumption that \( x > 0 \).
Then changing back from \( u \) to \( y \) we have
\[
\sin^{-1}(y/x) = \ln(x) + C.
\]
With the initial condition \( y(1) = 0 \) we find \( C = 0 \) so the solution is
\[
\sin^{-1}(y/x) = \ln(x).
\]

(4) [25] Find the most general family of solutions to the equation
\[
(3x^2 + y) \, dx + (x^2y - x) \, dy = 0.
\]
We can solve this using integrating factors for exact equations. Letting \( M = 3x^2 + y \) and \( N = x^2y - x \) we have that \( M_y = 1 \) and \( N_x = 2xy - 1 \). Then
\[
\frac{M_y - N_x}{N} = \frac{2 - 2xy}{x^2y - x} = \frac{-2(2xy - 1)}{x(yx - 1)} = \frac{-2}{x}.
\]
Consequently, we can find our integrating factor as
\[
\mu(x) = e^{\int \frac{-2}{x} \, dx} = x^{-2}.
\]
After multiplying through my \( \mu \) we have the equation
\[
(3 + x^{-2}y) \, dx + (y - x^{-1}) \, dy = 0.
\]
This equation is exact. Then we look for solutions of the form \( F(x, y) = C \). We know that \( F_x = 3 + x^{-2}y \) and \( F_y = y - x^{-1} \).
Integrating \( F_x \) we get
\[
F = 3x - x^{-1}y + g(y).
\]
Taking the \( y \) partial derivative we have
\[
F_y = x^{-1}y + g'(y)
\]
Since \( F_y = y - x^{-1} \)
\[
y - x^{-1} = x^{-1}y + g'(y)
\]
\[
g'(y) = y
\]
\[
g(y) = \frac{y^2}{2}
\]
Then \( F(x, y) = 3x - x^{-1}y + \frac{y^2}{2} \) and solutions are given implicitly by
\[
3x - x^{-1}y + \frac{y^2}{2} = C.
\]