MHF 3202, Exam I, Fall 2015

Name:_____

Turn in all relevant work with final answers circled on separate sheets. Full work is required for full credit.

(1) Analyze the validity of the following arguments

(a) If I won the lottery I would have a yacht.
I did not win the lottery.
Therefore, I do not have a yacht.

(b) $P \wedge \neg Q$.
$\neg(P \to \neg Q)$.
$\therefore R$.

Recall that an argument is sound iff it is valid and the premises are true. Which of the above arguments can be sound?

Let L stand for the sentence "I won the lottery" and S stand for "I have a yacht." Then the argument takes the form
$L \to Y$.
$\neg L$.
$\therefore \neg Y$.

If L is false and Y is false, then both premises are false but the conclusion is true. Therefore the argument is **invalid** and consequently **unsound**.

For part b, obseve that the second premise can be rewritten as $P \wedge Q$. So the two premises are contradictory, that is they can never be simultaneously true. This can also be seen by making the truth table and observing that no row has both premises true. Recall that validity means that if the premises are true then the conclusion is true. Since the premises are never true, that implication is always true, so the argument is **valid**. But since the premises are contradictory, the argument is **unsound**.

(2) The joint denial logical connective (nor) is denoted by $\downarrow$ and defined by the following truth table.

P	Q	$P \downarrow Q$
T	T	F
T	F	F
F	T	F
F	F	T

(a) Express $P \downarrow Q$ in terms of $\wedge, \vee,$ and, $\neg$.
(b) Express $\neg P$ in terms only of $\downarrow$.
(c) Express $P \wedge Q$ in terms only of $\downarrow$.

You can verify by truth table that $P \downarrow Q \leftrightarrow \neg(P \vee Q)$ and $\neg P \leftrightarrow (P \downarrow P)$. Then since $(P \wedge Q) \leftrightarrow \neg(\neg P \vee \neg Q)$ we can use the previous parts to show that it can be written $(P \downarrow P) \downarrow (Q \downarrow Q)$.

(3) For this question, you should use $C(f, x)$(or $D(f, x)$) to stand for "function f is continuous (or differentiable) at the point x." Express each of the following statements in logical notation.

(a) Whenever a function is differentiable at a point, it is also continuous at that point.

$$\forall f \forall x (D(f, x) \to C(f, x)).$$

(b) f is continuous at all but two real numbers.

$\exists x \exists y(\neg C(f,x) \wedge \neg C(f,y) \wedge (\forall z(z = x \vee z = y \vee C(f,z)))$.

Note that the last term of the statement could be written

$$\forall z((z \neq x \wedge z \neq y) \rightarrow C(f,z)).$$

(c) If a funtion is continous at every point in a closed interval $[a,b]$, it has a maximum value on that interval. If f takes its maximum value at some point z we mean that $\forall x(f(x) \leq f(z)$. So the above becomes

$$\forall f \forall a \forall b($$

$(\forall x \in [a,b](C(f,x))) \rightarrow (\exists z \in [a,b](\forall x \in [a,b] f(x) \leq f(z)))$.

Notice that it is $\forall a \forall b$ because the claim should hold for any interval $[a,b]$.

* If values of x are chosen sufficiently close to 0, the corresponding values of $f(x)$ are within 1 of 0.

** $\lim_{x \to x_0} f(x) = L$ if and only if values of $f(x)$ can be made arbitrarily close to L by choosing x-values are sufficiently close to x_0

(4) Define $D_i = \{ni \mid n \in \mathbb{N}\}$ for every $i \in \mathbb{N}$.

(a) Write out the definition of D_i in the form $\{x \mid \cdots\}$ and in idiomatic English.

$D_i = \{x \mid \exists n \in \mathbb{N}(x = ni)\}$. In plain English this means D_i is the set of all natural number multiples of i.

(b) Express $x \in \bigcup_{i \in \mathbb{N}} D_i$ and $x \in \bigcap_{i \in \mathbb{N}} D_i$ in logical notation and in idiomatic English.

$x \in \bigcup_{i \in \mathbb{N}} D_i \leftrightarrow \exists i \in \mathbb{N}(x \in D_i)$. This means that x is a natural number multiple of some natural number.

$x \in \bigcap_{i \in \mathbb{N}} D_i \leftrightarrow \forall i \in \mathbb{N}(x \in D_i)$. This means that x is a natural number multiple of every natural number.

(c) What is $\bigcup_{i \in \mathbb{N}} D_i$? $\bigcap_{i \in \mathbb{N}} D_i$?

Since $i \in D_i$ it must be that $\bigcup_{i \in \mathbb{N}} D_i = \mathbb{N}$. Similarly, since $i \notin_{i+1}$ unless $i = 0$ we have that $\bigcap_{i \in \mathbb{N}} D_i = \{0\}$.

(5) Use $C(f,x)$ and $D(f,x)$ as in the previous problem. We will also use $\mathbb{N}^+$ to stand for the set of positive natural numbers. Translate each of the below statements into idiomatic English. Then negate the statements, expressing the result without using $\neg$ (but perhaps using $\neq, \notin$).

(a) $\forall n \in \mathbb{N}(n > 2 \rightarrow \forall a \in \mathbb{N}^+ \forall b \in \mathbb{N}^+ \forall c \in \mathbb{N}^+ (a^n + b^n \neq c^n))$

If $n > 2$ there are no natural numbers $a, b, c > 2$ which satisfy the equation $a^n + b^n = c^n$. Recall that $\neg(P \rightarrow Q) \leftrightarrow (P \wedge \neg Q)$. In the above you can set $P = N > 2$ and $Q = \forall a \in \mathbb{N}^+ \forall b \in \mathbb{N}^+ \forall c \in \mathbb{N}^+ (a^n + b^n \neq c^n))$. Then the negation becomes

$\exists n \in \mathbb{N}(n > 2 \wedge \exists a \in \mathbb{N}^+ b \in \mathbb{N}^+ c \in \mathbb{N}^+ (a^n + b^n = c^n))$.

(b) $\forall a \forall b \forall f(\forall x \in [a,b](C(f,x)) \wedge \forall x \in (a,b)(D(f,x))) \rightarrow$
$(\exists c \in (a,b)(f'(c) = \frac{f(b)-f(a)}{b-a}))$.

If f is continuous on an interval $[a, b]$ and differentiable on (a, b) then the slope of the line joining a and b is equal the the function's derivative at some point in (a, b).

* What are the names of the above theorems? They are **Fermat's Last Theorem** and the **Mean Value Theorem**.