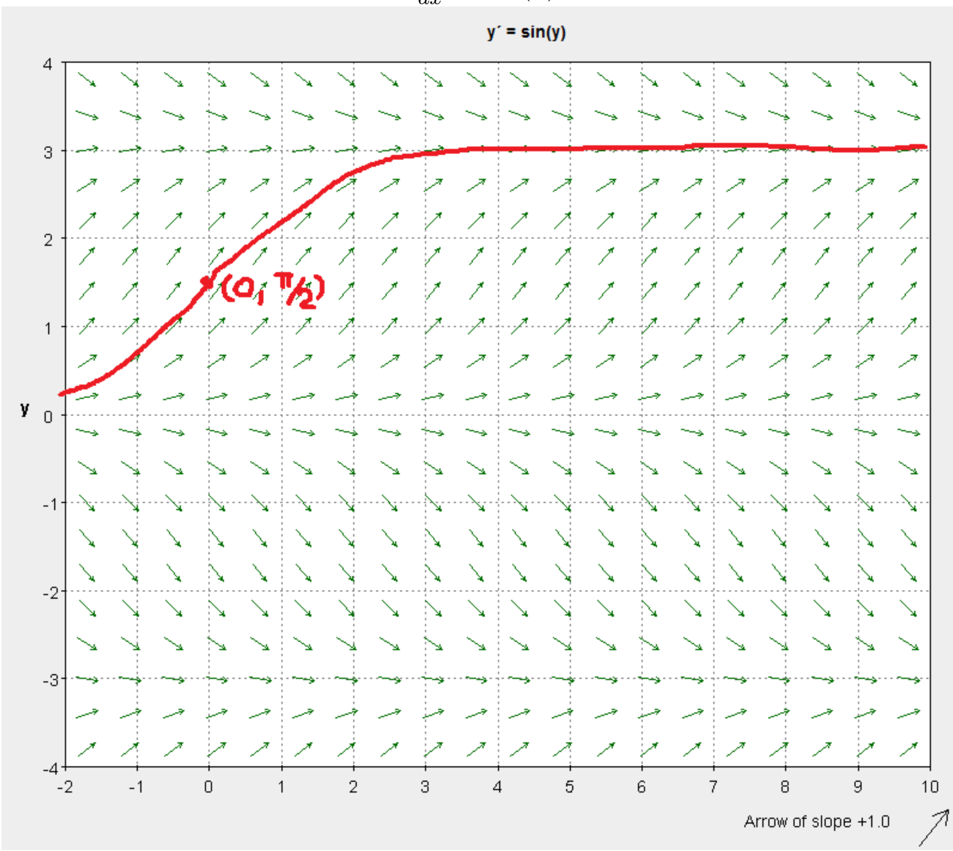
MAP 2302, Exam I Sample, Fall 2014

Name:______________________

Student signature:______________________

Write final answers on this sheet. Turn in all relevant work on separate sheets.

(1) The direction field for the ODE $\frac{dy}{dx} = \sin(y)$ is shown below.



Answer the following questions about the ODE and its direction field.

(a) What are all of the constant solutions of the ODE?
We test the functions $y = C$ as a solutions to the ODE. Every such function has $\frac{dy}{dx} = 0$, so plugging them into the ODE yields the relation $0 = \sin(C)$. It follows that $y = C$ is a solution whenever $C = k\pi$ for any integer k.

(b) Sketch the solution to the IVP with initial condition $y(0) = \pi/2$.
See graph.

(c) Find a solution for the IVP with initial condition $y(0) = \pi/2$.
Use separation of variables.

$$\frac{dy}{dx} = \sin(y)$$

$$\frac{dy}{\sin(y)} = dx$$

$$-\ln|\csc(y) + \cot(y)| = x + C$$

Using the initial condition $y = \pi/2, x = 0$ gives

$$\begin{aligned} -\ln|1+0| &= C \\ 0 &= C \end{aligned}$$

And so the solution to the IVP is given implicitely by

$$-\ln|\csc(y)+\cot(y)| = x$$

(2) Show that the equation $\frac{dy}{dx} = \frac{x+y}{x-y}$ is homogeneous. Solve the ODE using the substitution $v = \frac{y}{x}$.
We have $f(x,y) = \frac{x+y}{x-y}$ so $f(tx,ty) = \frac{tx+ty}{tx-ty} = \frac{t(x+y)}{t(x-y)} = f(x,y)$. So the equation is homogeneous. Using $v = y/x$ we have $vx = y$ and $v + x\frac{dv}{dx} = \frac{dy}{dx}$. This transforms the ODE into

$$\begin{aligned} v + x\frac{dv}{dx} &= \frac{x+vx}{x-vx} \\ x\frac{dv}{dx} &= \frac{x+vx}{x-vx} - v\frac{x-vx}{x-vx} \\ x\frac{dv}{dx} &= \frac{x+v^2x}{x-vx} \end{aligned}$$

This is a separable equation

$$\begin{aligned} x\frac{dv}{dx} &= \frac{1+v^2}{1-v} \\ \frac{1-v}{1+v^2}dv &= dx \\ \int \frac{1}{1+v^2}\,dv - \int \frac{v}{1+v^2}\,dv &= \int \frac{1}{x}\,dx \\ \arctan(v) - \frac{1}{2}\ln\left(1+v^2\right) &= \ln|x| + C \end{aligned}$$

Back substituting $v = y/x$ gives

$$\arctan\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(1 + \left(\frac{y}{x}\right)^2\right) = \ln|x| + C$$

(3) Find an integrating factor to make the following equation exact. Then find the most general solution for the equation.

$$(2y^2 + 2y + 4x^2)\,dx + (2xy + x)\,dy = 0$$

Call $M = 2y^2 + 2y + 4x^2, N = 2xy + x$. Then $\frac{\partial M}{\partial y} = 4y + 2$ and $\frac{\partial N}{\partial x} = 2y + 1$ so the equation is not exact. We first look for an integrating factor $\mu(x)$.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y-1}{x(2y+1)} = \frac{1}{x}.$$

Since this is a function of x, we have $\mu(x) = e^{\int (1/x)\,dx} = x$. After multiplying through we get

$$(2xy^2 + 2xy + 4x^3)\,dx + (2x^2y + x^2)\,dy = 0.$$

Now we have $M = 2xy^2 + 2xy + 4x^3, N = 2x^2y + x^2$ and can follow the method of exact equations.

$$F(x,y) = \int (2xy^2 + 2xy + 4x^3)\,dx = x^2y^2 + x^2y + x^4 + g(y)$$

$$2x^2y + x^2 = \frac{\partial F}{\partial y} = 2xy^2 + x^2 = g'(y)$$

Consequently $g'(y) = y$ so $g(y) = 0$ and solutions are given by $x^2y^2 + x^2y + x^4 = C$.

(4) Find the general solution to

$$\frac{dy}{dx} = x^2e^{-4x} - 4y$$

This linear equation can be put in standard form

$$\frac{dy}{dx} + 4y = x^2e^{-4x}$$

taking $\mu(x) = e^{4x}$ as an integrating factor gives

$$e^{4x}\frac{dy}{dx} + 4e^{4x}y = x^2$$

$$\frac{d}{dx}\left(ye^{4x}\right) = x^2$$

$$ye^{4x} = \frac{x^3}{3} + C$$

$$y = \frac{x^3}{3}e^{-4x} + Ce^{-4x}$$