Answer the following questions about the ODE and its direction field.

(a) What are all of the constant solutions of the ODE?

We test the functions $y = C$ as solutions to the ODE. Every such function has $\frac{dy}{dx} = 0$, so plugging them into the ODE yields the relation $0 = \sin(C)$. It follows that $y = C$ is a solution whenever $C = k\pi$ for any integer $k$.

(b) Sketch the solution to the IVP with initial condition $y(0) = \pi/2$.

See graph.

(c) Find a solution for the IVP with initial condition $y(0) = \pi/2$.

Use separation of variables.

\[
\frac{dy}{dx} = \sin(y) \\
\frac{dy}{\sin(y)} = dx \\
-\ln|\csc(y) + \cot(y)| = x + C
\]
Using the initial condition \( y = \pi/2, x = 0 \) gives
\[
-\ln|1 + 0| = C
\]
\[
0 = C
\]
And so the solution to the IVP is given implicitly by
\[
-\ln|\csc(y) + \cot(y)| = x
\]

(2) Show that the equation \( \frac{dy}{dx} = \frac{x+y}{x-y} \) is homogeneous. Solve the ODE using the substitution \( v = \frac{y}{x} \).
We have \( f(x, y) = \frac{x+y}{x-y} \) so \( f(tx, ty) = \frac{tx+ty}{tx-ty} = \frac{t(x+y)}{t(x-y)} = f(x, y) \). So the equation is homogeneous. Using \( v = y/x \) we have \( vx = y \) and \( v + x \frac{dv}{dx} = \frac{dy}{dx} \). This transforms the ODE into
\[
v + x \frac{dv}{dx} = \frac{x + vx}{x - vx}
\]
\[
x \frac{dv}{dx} = \frac{x + vx}{x - vx} - \frac{x - vx}{x - vx} = \frac{x + v^2 x}{x - vx}
\]
This is a separable equation
\[
x \frac{dv}{dx} = \frac{1 + v^2}{1 - v}
\]
\[
\int \frac{1}{1 + v^2} dv - \int \frac{v}{1 + v^2} dv = \int \frac{1}{x} dx
\]
\[
\arctan(v) - \frac{1}{2} \ln(1 + v^2) = \ln |x| + C
\]
Back substituting \( v = y/x \) gives
\[
\arctan \left( \frac{y}{x} \right) - \frac{1}{2} \ln \left( 1 + \left( \frac{y}{x} \right)^2 \right) = \ln |x| + C
\]

(3) Find an integrating factor to make the following equation exact. Then find the most general solution for the equation.
\[
(2y^2 + 2y + 4x^2) \, dx + (2xy + x) \, dy = 0
\]
Call \( M = 2y^2 + 2y + 4x^2, N = 2xy + x \). Then \( \frac{\partial M}{\partial y} = 4y + 2 \) and \( \frac{\partial N}{\partial x} = 2y + 1 \) so the equation is not exact. We first look for an integrating factor \( \mu(x) \).
\[
\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{2y - 1}{x(2y + 1)} = \frac{1}{x}.
\]
Since this is a function of $x$, we have $\mu(x) = e^{f(1/x)} dx = x$. After multiplying through we get

$$\int (2xy^2 + 2xy + 4x^3) \, dx + (2x^2y + x^2) \, dy = 0.$$ 

Now we have $M = 2xy^2 + 2xy + 4x^3, N = 2x^2y + x^2$ and can follow the method of exact equations.

$$F(x, y) = \int (2xy^2 + 2xy + 4x^3) \, dx = x^2y^2 + x^2y + x^4 + g(y)$$

$$2x^2y + x^2 = \frac{\partial F}{\partial y} = 2xy^2 + x^2 = g'(y)$$

Consequently $g'(y) = y$ so $g(y) = 0$ and solutions are given by $x^2y^2 + x^2y + x^4 = C$.

(4) Find the general solution to

$$\frac{dy}{dx} = x^2e^{-4x} - 4y$$

This linear equation can be put in standard form

$$\frac{dy}{dx} + 4y = x^2e^{-4x}$$

taking $\mu(x) = e^{4x}$ as an integrating factor gives

$$e^{4x} \frac{dy}{dx} + 4e^{4x}y = x^2$$

$$\frac{d}{dx} (ye^{4x}) = x^2$$

$$ye^{4x} = \frac{x^3}{3} + C$$

$$y = \frac{x^3}{3}e^{-4x} + Ce^{-4x}$$