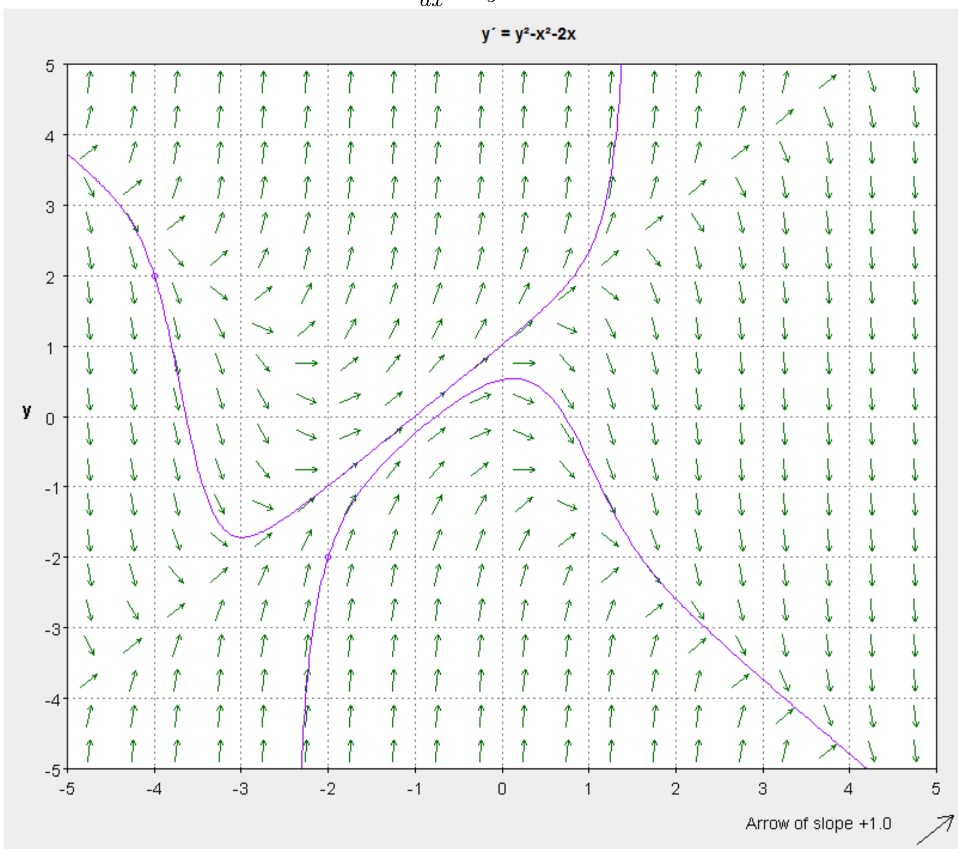
MAP 2302, Exam I, Spring 2015

Name:________________________

Student signature:________________________

Write final answers on this sheet. Turn in all relevant work on separate sheets.

(1) The direction field for the ODE $\frac{dy}{dx} = y^2 - x^2 - 2x$ is shown below.



Answer the following questions about the ODE and its direction field.

(a) [8] What are all of the linear solutions($y = ax + b$ for some a, b) of the ODE?
Plug $y = ax + b$ into the ODE. We get

$$a = (ax + b)^2 - x^2 - 2x = (a - 1)x^2 + (2ab - 2)x + b^2.$$

Equating coefficients on the left and right side gives the relations $a^2 = 1, ab = 1$, and $b^2 = a$. From the last we get that $a > 0$, so that $a = 1$ and not -1. Then $ab = 1$ gives that $b = 1$, so the only linear solution is $y = x + 1$.

(b) [6] Sketch the solutions to the IVPs with initial conditions $y(-4) = 2$ and $y(-2) = -2$.
Shown above.

(c) [6] Is there a solution to the ODE which satisfies both $y(-3) = 0$ and $y(-2) = -2$? Justify your answer.
Evidently, such a solution would cross the solution $y = x + 1$ as each condition is on opposite sides of the line. Since $f(x, y) =$

$y^2 - x^2 - 2x$ is continuous everywhere and so is $\frac{\partial f}{\partial y} = 2y$, the uniqueness theorem says that no two solutions cross. So this is impossible.

(2) [15] Use separation of variables to solve the IVP

$$\frac{dy}{dx} = \frac{\sin(x)}{\cos(y)} \quad , y(0) = \pi.$$

We get

$$\begin{aligned} \cos(y)\, dy &= \sin(x)\, dx \\ \sin(y) &= -\cos(x) + C \end{aligned}$$

Using $y(0) = \pi$ we get

$$\begin{aligned} 0 &= -1 + C \\ C &= 1 \end{aligned}$$

So the solution is

$$\sin(y) = -\cos(x) + 1$$

(3) [15] Find the general solution to

$$\frac{dy}{dx} + \frac{3}{x}y + 2 = 3x.$$

This is a linear equation. We rewrite in standard form

$$\frac{dy}{dx} + \frac{3}{x}y = 3x - 2$$

An integrating factor is $\mu(x) = e^{\int (3/x)\, dx} = x^3$

$$\begin{aligned} \frac{d}{dx}(x^3 y) &= 3x^4 + 2x^3 \\ x^3 y &= \frac{3}{5}x^5 + \frac{1}{2}x^4 + C \\ y &= \frac{3}{5}x^2 + \frac{1}{2}x + Cx^{-3} \end{aligned}$$

(4) [20] Show that the equation $\frac{dy}{dx} = \frac{x^3+xy^2-x}{y}$ is **not** homogeneous. Solve the ODE using the substitution $v = x^2 + y^2$.
Here we have $f(x,y) = \frac{x^3+xy^2-x}{y}$. So then

$$f(tx,ty) = \frac{t^3x^3 + t^3xy^2 - tx}{ty} = \frac{t^2x^3 + t^2xy^2 - x}{y} \neq f(x,y).$$

So the ODE Is not homogeneous. To use the substitution $v = x^2+y^2$, we find $\frac{dv}{dx} = 2x + 2y\frac{dy}{dx}$. We will use this in the form $y\frac{dy}{dx} = \frac{1}{2}\frac{dv}{dx} - x$

Applying these rules we obtain

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x^3 + xy^2 - x}{y} \\
y\frac{dy}{dx} &= x^3 + xy^2 - x = x(x^2 + y^2) - x \\
\frac{1}{2}\frac{dv}{dx} - x &= xv - x \\
\frac{dv}{dx} &= 2xv
\end{aligned}$$

We can solve the above separable ODE.

$$\begin{aligned}
\frac{dv}{v} &= 2xdx \\
\ln|v| &= x^2 + C \\
\ln(x^2 + y^2) &= x^2 + C
\end{aligned}$$

Note that since $x^2+y^2 \geq 0$ we are allowed to drop the absolute value in the last line.

(5) [30] Show that there is no integrating factor which depends only on x for the ODE (10)

$$(3y + 2xy^2)\,dx + (x + 2x^2y)\,dy = 0$$

Then use the integrating factor $\mu(x, y) = \frac{1}{xy}$ to solve the ODE (20). Set $M = 3y + 2xy^2, N = x + 2x^2y$. Then an integrating factor depending only on x exists if and only if the function

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

depends on x alone. We can compute

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3 + 4xy - (1 + 4xy)}{x + 2x^2y} = \frac{4}{x + 2x^2y}.$$

Evidently, this depends on both x and y, so there is no such integrating factor.

Using $\mu(x, y) = \frac{1}{xy}$, we multiply through to obtain the ODE

$$(\frac{3}{x} + 2y)\,dx + (\frac{1}{y} + 2x)\,dy = 0$$

Now set $M = (\frac{3}{x} + 2y), N = (\frac{1}{y} + 2x)$. We can see that this equation is exact by noting $\frac{\partial M}{\partial y} = 2 = \frac{\partial N}{\partial x}$. Then

$$\begin{aligned}
F(x, y) &= \int M\,dx = \int (\frac{3}{x} + 2y)\,dx + g(y) \\
F(x, y) &= 3\ln|x| + 2xy + g(y)
\end{aligned}$$

Taking $\frac{\partial}{\partial y}$ we get

$$\begin{aligned} \frac{1}{y} + 2x = N = \frac{\partial F}{\partial y} &= 2x + g'(y) \\ g'(y) &= \frac{1}{y} \\ g(y) &= \ln|y| \end{aligned}$$

Then $F(x, y) = 3\ln|x| + 2xy + \ln|y|$ and solutions are given by $3\ln|x| + 2xy + \ln|y| = C$.