Answer the following questions about the ODE and its direction field.

(a) [8] What are all of the linear solutions \( y = ax + b \) for some \( a, b \) of the ODE?

Plug \( y = ax + b \) into the ODE. We get

\[ a = (ax + b)^2 - x^2 - 2x = (a - 1)x^2 + (2ab - 2)x + b^2. \]

Equating coefficients on the left and right side gives the relations \( a^2 = 1 \), \( ab = 1 \), and \( b^2 = a \). From the last we get that \( a > 0 \), so that \( a = 1 \) and not \(-1\). Then \( ab = 1 \) gives that \( b = 1 \), so the only linear solution is \( y = x + 1 \).

(b) [6] Sketch the solutions to the IVPs with initial conditions \( y(-4) = 2 \) and \( y(-2) = -2 \).

Shown above.

(c) [6] Is there a solution to the ODE which satisfies both \( y(-3) = 0 \) and \( y(-2) = -2 \)? Justify your answer.

Evidently, such a solution would cross the solution \( y = x + 1 \) as each condition is on opposite sides of the line. Since \( f(x, y) = \)
\[ y^2 - x^2 - 2x \] is continuous everywhere and so is \( \frac{\partial f}{\partial y} = 2y \), the uniqueness theorem says that no two solutions cross. So this is impossible.

(2) [15] Use separation of variables to solve the IVP

\[
\frac{dy}{dx} = \frac{\sin(x)}{\cos(y)}, \quad y(0) = \pi.
\]

We get

\[
\cos(y)\,dy = \sin(x)\,dx \\
\sin(y) = -\cos(x) + C
\]

Using \( y(0) = \pi \) we get

\[
0 = -1 + C \\
C = 1
\]

So the solution is

\[
\sin(y) = -\cos(x) + 1
\]

(3) [15] Find the general solution to

\[
\frac{dy}{dx} + \frac{3}{x}y + 2 = 3x.
\]

This is a linear equation. We rewrite in standard form

\[
\frac{dy}{dx} + \frac{3}{x}y = 3x - 2
\]

An integrating factor is \( \mu(x) = e^{\int \frac{3}{x} \, dx} = x^3 \)

\[
\frac{d}{dx}(x^3 y) = 3x^4 + 2x^3 \\
x^3 y = \frac{3}{5} x^5 + \frac{1}{2} x^4 + C \\
y = \frac{3}{5} x^2 + \frac{1}{2} x + C x^{-3}
\]

(4) [20] Show that the equation \( \frac{dy}{dx} = \frac{x^3 + xy^2 - x}{y} \) is not homogeneous.

Solve the ODE using the substitution \( v = x^2 + y^2 \).

Here we have \( f(x, y) = \frac{x^3 + xy^2 - x}{y} \). So then

\[
f(tx, ty) = \frac{t^3x^3 + t^3xy^2 - tx}{ty} = t^2x^3 + \frac{t^2xy^2 - x}{y} \neq f(x, y).
\]

So the ODE is not homogeneous. To use the substitution \( v = x^2 + y^2 \), we find \( \frac{dv}{dx} = 2x + 2y \frac{dy}{dx} \). We will use this in the form \( y \frac{dy}{dx} = \frac{1}{2} \frac{dv}{dx} - x \).
Applying these rules we obtain
\[
\frac{dy}{dx} = \frac{x^3 + xy^2 - x}{y}
\]
\[
y \frac{dy}{dx} = x^3 + xy^2 - x = x(x^2 + y^2) - x
\]
\[
\frac{1}{2} \frac{dv}{dx} - x = xv - x
\]
\[
\frac{dv}{dx} = 2xv
\]

We can solve the above separable ODE.
\[
\frac{dv}{v} = 2xdx
\]
\[
\ln |v| = x^2 + C
\]
\[
\ln(x^2 + y^2) = x^2 + C
\]

Note that since \(x^2 + y^2 \geq 0\) we are allowed to drop the absolute value in the last line.

(5) [30] Show that there is no integrating factor which depends only on \(x\) for the ODE (10)

\[(3y + 2xy^2) \, dx + (x + 2x^2y) \, dy = 0\]

Then use the integrating factor \(\mu(x, y) = \frac{1}{xy}\) to solve the ODE (20).

Set \(M = 3y + 2xy^2, N = x + 2x^2y\). Then an integrating factor depending only on \(x\) exists if and only if the function

\[
\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}
\]

depends on \(x\) alone. We can compute

\[
\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3 + 4xy - (1 + 4xy)}{x + 2x^2y} = \frac{4}{x + 2x^2y}.
\]

Evidently, this depends on both \(x\) and \(y\), so there is no such integrating factor.

Using \(\mu(x, y) = \frac{1}{xy}\), we multiply through to obtain the ODE

\[
\left(\frac{3}{x} + 2y\right) \, dx + \left(\frac{1}{y} + 2x\right) \, dy = 0
\]

Now set \(M = (\frac{3}{x} + 2y), N = (\frac{1}{y} + 2x)\). We can see that this equation is exact by noting \(\frac{\partial M}{\partial y} = 2 = \frac{\partial N}{\partial x}\). Then

\[
F(x, y) = \int M \, dx = \int \left(\frac{3}{x} + 2y\right) \, dx + g(y)
\]
\[
F(x, y) = 3 \ln |x| + 2xy + g(y)
\]
Taking $\frac{\partial}{\partial y}$ we get

$$\frac{1}{y} + 2x = N = \frac{\partial F}{\partial y} = 2x + g'(y)$$

$$g'(y) = \frac{1}{y}$$

$$g(y) = \ln|y|$$

Then $F(x, y) = 3\ln|x| + 2xy + \ln|y|$ and solutions are given by $3\ln|x| + 2xy + \ln|y| = C$. 