MAP 2302, Exam II, Spring 2015

Name:________________________

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Write final answers on this sheet. Turn in all relevant work on separate sheets.

(1) Use the method of undetermined coefficients to find the form of a particular solution y_p to the following ODEs. **Do not solve the equation!**

(a) $y'' - 2y' + 2y = e^t$
The roots of the characteristic polynomial $r^2 - 2r + 2$ are $r = 1 \pm i$. Looking at the right hand side of the ODE, we set $y_p = t^s A_0 e^t$. Since 1 is not a root of the characteristic polynormial, we set $s = 0$ so $y_p = A_0 e^t$.

(b) $y'' - 2y' + 2y = te^t \cos(t) + t^2 e^t \sin(t)$
The method says to use the form

$$y_p = t^s \left(\left(A_2 t^2 + A_1 t + A_0\right) e^t \sin(t) + \left(B_2 t^2 + B_1 t + B_0\right) e^t \cos(t) \right).$$

Since $1 + i$ is a root of the characteristic polynomial, we choose $s = 1$ and have

$$y_p = t \left(\left(A_2 t^2 + A_1 t + A_0\right) e^t \sin(t) + \left(B_2 t^2 + B_1 t + B_0\right) e^t \cos(t) \right).$$

(c) $y'' + 4y' + 4y = e^{-2t}$
For this equation, the characteristic polynomial has a double root $r = -2$. We set $y_p = t^s(A_0)e^{-2t}$ Since -2 is a double root, set $s = 2$ so $y_p = t^2(A_0)e^{-2t}$.

(d) $y'' + 4y' + 4y = e^{-2t} + e^t$
By the superposition principle, we can solve separately for e^{-2t} and e^t. We previously found a form for when the right hand side is e^{-2t}. Following the same argument, the form for e^t is $A_0 e^t$. Adding them together (and being sure not to reuse A_0) we have $y_p = t^2 A_0 e^{-2t} + B_0 e^t$.

(2) Find the general solution to the following ODE for $t < 0$.

$$t^2 y'' + 4ty' + 2y = \sin(t)$$

This is a Cauchy-Euler equation which can be written in standard form as

$$y'' + \frac{4}{t} y' + \frac{2}{t^2} y = \frac{\sin(t)}{t^2}.$$

Recall that $y = y_p + y_h$ is the general solution where y_p is one particular solution to the ODE and y_h is the general solution to the associated homogeneous ODE. Since we need y_h to find y_p we will start by finding y_h. The characteristic polynomial for this Cauchy-Euler equation is $r^2 + 3r + 2$ which has roots $r = -1, -2$. So

$$y_h = k_1(-t)^{-1} + k_2(-t)^{-2}$$

. We will call $y_1 = (-t)^{-1} = -1/t, y_2 = (-t)^{-2} = 1/t^2$. The Wronskian (which we need for variation of parameters) is computed to be

$$\begin{aligned} W[y_1, y_2] &= y_1 y_2' - y_1' y_2 \\ &= (-t)^{-1}(2(-t)^{-3}) - (-t)^{-2}(-t)^{-2} \\ &= (-t)^{-4} \\ &= t^{-4}. \end{aligned}$$

Then by variation of parameters we have $y_p = v_1 y_1 + v_2 y_2$ where

$$\begin{aligned} v_1 &= \int \frac{-g y_2}{W[y_1, y_2]}\, dt = -\int \frac{t^{-2}\sin(t) t^{-2}}{t^{-4}}\, dt \\ v_1 &= -\int \sin(t)\, dt = \cos(t) \end{aligned}$$

and,

$$\begin{aligned} v_2 &= \int \frac{g y_1}{W[y_1, y_2]}\, dt = \int \frac{t^{-2}\sin(t)(-t^{-1})}{t^{-4}}\, dt \\ v_2 &= -t\sin(t)\, dt = t\cos(t) - \sin(t) \end{aligned}$$

Then $y_p = -t^{-1}\cos(t) + t^{-1}\cos(t) - t^{-2}\sin(t) = -t^{-2}\sin(t)$ and

$$y = -t^{-2}\sin(t) + k_1(-t)^{-1} + k_2(-t)^{-2}$$

is the general solution.

(3) If y_1, y_2 are solutions to $y'' + t^2 y' + e^t y = 0$ on $(-\infty, \infty)$ can $W[y_1, y_2](t) = t$ be their Wronskian?
No. The functions t^2, e^t are continuous on $(-\infty, \infty)$. Then the Wronskian of two solutions on $(-\infty, \infty)$ is either identically zero or never equal to zero. However, t is zero only for $t = 0$.

(4) Verify that $y_1(t) = t$ is a solution to

$$(1 - t^2)y'' - 2ty' + 2y = 0.$$

Then find the general solution to that ODE for $t > 1$.
Call $y_1 = t$. Then $y_1' = 1, y_1'' = 0$. Plugging this into the above ODE gives

$$(1 - t^2)(0) - 2t(1) + 2t = 0.$$

So $y_1 = t$ is a solution. To find the second linearly independent solution, we use reduction of order. First we put the equation in standard form

$$y'' - \frac{2t}{1-t^2}y' + \frac{2}{1-t^2}y = 0.$$

Remember that discontinuities at $t = \pm 1$ force us to look for solutions either on $(-\infty, 1), (-1, 1)$, or $(1, \infty)$. The problem specifies that we are searching on $(1, \infty)$. Now the reduction of order formula gives us

$$y_2 = y_1 \int \frac{e^{-\int p(t)\,dt}}{(y_1)^2}\,dt$$
$$y_2 = t \int \frac{e^{-\int \frac{-2t}{1-t^2}\,dt}}{t^2}\,dt$$
$$y_2 = t \int \frac{e^{-\ln|1-t^2|}}{t^2}\,dt$$

Since $t > 1$ we have $|1-t^2| = t^2 - 1$ and

$$y_2 = t \int \frac{1}{t^2(1-t^2)}$$

This is where you do partial fractions to get

$$y_2 = -1 + \frac{t}{2}\ln\left(\frac{t+1}{t-1}\right)$$

Then the general solutions is

$$y = k_1 t + k_2\left(-1 + \frac{t}{2}\ln\left(\frac{t+1}{t-1}\right)\right).$$

(5) Solve the IVP

$$y'' + 5y' + 6y = \sin(t) \quad, y(0) = 1, y'(0) = -1.$$

The general solution is $y = y_h + y_p$. The roots of the characteristic polynomial are $r = -2, -3$ so $y_h = k_1 e^{-2t} + k_2 e^{-3t}$ is the general solution. To find y_p, we use the method of undetermined coefficients. The form is

$$y_p = A\sin(t) + B\cos(t)$$

and so,

$$y_p' = A\cos(t) - B\sin(t)$$
$$y_p'' = -A\sin(t) - B\cos(t)$$

Plugging into the ODE gives

$$\sin(t)\,[-A - 5B + 6A] + \cos(t)\,[-B + 5A + 6B] = \sin(t).$$

By equating coefficients on the left and right, this gives rise to the linear system

$$5A - 5B = 1$$
$$5A + 5B = 0$$

We can solve this to obtain $A = 1/10, B = -1/10$. So the general solution is given by

$$y = \frac{1}{10}\sin(t) - \frac{1}{10}\cos(t) + k_1 e^{-2t} + k_2 e^{-3t}$$

and its derivative is

$$y' = \frac{1}{10}\cos(t) + \frac{1}{10}\sin(t) - 2k_1 e^{-2t} - 3k_2 e^{-3t}$$

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Now we adjust k_1, k_2 to match the initial conditions. Plugging in $y(0) = 1$ we get $1 = -\frac{1}{10} + k_1 + k_2$. Plugging in $y'(0) = -1$ gives $-1 = \frac{1}{10} - 2k_1 - 3k_2$. We solve this linear system to get $k_1 = 11/5, k_2 = -11/10$. So the solution to the IVP is

$$y = \frac{1}{10}\sin(t) - \frac{1}{10}\cos(t) + \frac{11}{5}e^{-2t} - \frac{11}{10}e^{-3t}.$$