MAP 2302, Exam II, Spring 2015

Write final answers on this sheet. Turn in all relevant work on separate sheets.

(1) Use the method of undetermined coefficients to find the form of a particular solution \( y_p \) to the following ODEs. Do not solve the equation!

(a) \( y'' - 2y' + 2y = e^t \)
   The roots of the characteristic polynomial \( r^2 - 2r + 2 \) are \( r = 1 \pm i \). Looking at the right hand side of the ODE, we set \( y_p = t^s A_0e^t \). Since 1 is not a root of the characteristic polynomial, we set \( s = 0 \) so \( y_p = A_0e^t \).

(b) \( y'' - 2y' + 2y = te^t \cos(t) + t^2 e^t \sin(t) \)
   The method says to use the form \( y_p = t^s ((A_2t^2 + A_1t + A_0) e^t \sin(t) + (B_2t^2 + B_1t + B_0) e^t \cos(t)) \).
   Since \( 1 + i \) is a root of the characteristic polynomial, we choose \( s = 1 \) and have \( y_p = t((A_2t^2 + A_1t + A_0) e^t \sin(t) + (B_2t^2 + B_1t + B_0) e^t \cos(t)) \).

(c) \( y'' + 4y' + 4y = e^{-2t} \)
   For this equation, the characteristic polynomial has a double root \( r = -2 \). We set \( y_p = t^s (A_0) e^{-2t} \) Since \( -2 \) is a double root, set \( s = 2 \) so \( y_p = t^2 (A_0) e^{-2t} \).

(d) \( y'' + 4y' + 4y = e^{-2t} + e^t \)
   By the superposition principle, we can solve separately for \( e^{-2t} \) and \( e^t \). We previously found a form for when the right hand side is \( e^{-2t} \). Following the same argument, the form for \( e^t \) is \( A_0e^t \). Adding them together (and being sure not to reuse \( A_0 \)) we have \( y_p = t^2 A_0 e^{-2t} + B_0 e^t \).

(2) Find the general solution to the following ODE for \( t < 0 \).

\[ t^2 y'' + 4ty' + 2y = \sin(t) \]

This is a Cauchy-Euler equation which can be written in standard form as

\[ y'' + \frac{4}{t}y' + \frac{2}{t^2}y = \frac{\sin(t)}{t^2} \]

Recall that \( y = y_p + y_h \) is the general solution where \( y_p \) is one particular solution to the ODE and \( y_h \) is the general solution to the associated homogeneous ODE. Since we need \( y_h \) to find \( y_p \) we will start by finding \( y_h \). The characteristic polynomial for this Cauchy-Euler equation is \( r^2 + 3r + 2 \) which has roots \( r = -1, -2 \). So

\[ y_h = k_1(-t)^{-1} + k_2(-t)^{-2} \]
We will call $y_1 = (-t)^{-1} = -1/t, y_2 = (-t)^{-2} = 1/t^2$. The Wronskian (which we need for variation of parameters) is computed to be

$$W[y_1, y_2] = y_1 y_2' - y_1' y_2$$

$$= (-t)^{-1} (2(-t)^{-3}) - (-t)^{-2}(-t)^{-2}$$

$$= (-t)^{-4}$$

$$= t^{-4}.$$ 

Then by variation of parameters we have $y_p = v_1 y_1 + v_2 y_2$ where

$$v_1 = \int \frac{-gy_2}{W[y_1, y_2]} \, dt = -\int \frac{t^{-2} \sin(t)t^{-2}}{t^{-4}} \, dt$$

$$v_1 = -\int \sin(t) \, dt = \cos(t)$$

and,

$$v_2 = \int \frac{gy_1}{W[y_1, y_2]} \, dt = -\int \frac{t^{-2} \sin(t)(-t^{-1})}{t^{-4}} \, dt$$

$$v_2 = t \sin(t) \, dt = t \cos(t) - \sin(t)$$

Then $y_p = -t^{-1} \cos(t) + t^{-1} \cos(t) - t^{-2} \sin(t) = -t^{-2} \sin(t)$ and

$$y = -t^{-2} \sin(t) + k_1(-t)^{-1} + k_2(-t)^{-2}$$

is the general solution.

(3) If $y_1, y_2$ are solutions to $y'' + t^2 y' + e^t y = 0$ on $(-\infty, \infty)$ can $W[y_1, y_2](t) = t$ be their Wronskian?

No. The functions $t^2, e^t$ are continuous on $(-\infty, \infty)$. Then the Wronskian of two solutions on $(-\infty, \infty)$ is either identically zero or never equal to zero. However, $t$ is zero only for $t = 0.$

(4) Verify that $y_1(t) = t$ is a solution to

$$(1 - t^2) y'' - 2ty' + 2y = 0.$$ 

Then find the general solution to that ODE for $t > 1$.

Call $y_1 = t$. Then $y_1' = 1, y_1'' = 0$. Plugging this into the above ODE gives

$$(1 - t^2)(0) - 2t(1) + 2t = 0.$$ 

So $y_1 = t$ is a solution. To find the second linearly independent solution, we use reduction of order. First we put the equation in
standard form
\[ y'' - \frac{2t}{1-t^2} y' + \frac{2}{1-t^2} y = 0. \]

Remember that discontinuities at \( t = \pm 1 \) force us to look for solutions either on \( (-\infty, 1) \), \( (-1, 1) \), or \( (1, \infty) \). The problem specifies that we are searching on \( (1, \infty) \). Now the reduction of order formula gives us

\[
y_2 = y_1 \int \frac{e^{-\int p(t) dt}}{(y_1)^2} \, dt \\
y_2 = t \int \frac{e^{-\int \frac{2t}{1-t^2} dt}}{t^2} \, dt \\
y_2 = t \int \frac{e^{-\ln|1-t^2|}}{t^2} \, dt
\]

Since \( t > 1 \) we have \( |1-t^2| = t^2 - 1 \) and

\[
y_2 = t \int \frac{1}{t^2(1-t^2)}
\]

This is where you do partial fractions to get

\[
y_2 = -1 + \frac{t}{2} \ln \left( \frac{t+1}{t-1} \right)
\]

Then the general solutions is

\[
y = k_1 t + k_2 \left( -1 + \frac{t}{2} \ln \left( \frac{t+1}{t-1} \right) \right)
\]

(5) Solve the IVP
\[ y'' + 5y' + 6y = \sin(t) \quad y(0) = 1, y'(0) = -1. \]

The general solution is \( y = y_h + y_p \). The roots of the characteristic polynomial are \( r = -2, -3 \) so \( y_h = k_1 e^{-2t} + k_2 e^{-3t} \) is the general solution. To find \( y_p \), we use the method of undetermined coefficients. The form is

\[
y_p = A \sin(t) + B \cos(t)
\]

and so,

\[
y_p' = A \cos(t) - B \sin(t) \\
y_p'' = -A \sin(t) - B \cos(t)
\]

Plugging into the ODE gives

\[
\sin(t) \left[ -A - 5B + 6A \right] + \cos(t) \left[ -B + 5A + 6B \right] = \sin(t).
\]
By equating coefficients on the left and right, this gives rise to the linear system

\[
\begin{align*}
5A - 5B &= 1 \\
5A + 5B &= 0
\end{align*}
\]

We can solve this to obtain \( A = 1/10, B = -1/10 \). So the general solution is given by

\[
y = \frac{1}{10} \sin(t) - \frac{1}{10} \cos(t) + k_1 e^{-2t} + k_2 e^{-3t}
\]

and its derivative is

\[
y' = \frac{1}{10} \cos(t) + \frac{1}{10} \sin(t) - 2k_1 e^{-2t} - 3k_2 e^{-3t}
\]

Now we adjust \( k_1, k_2 \) to match the initial conditions. Plugging in \( y(0) = 1 \) we get \( 1 = -\frac{1}{10} + k_1 + k_2 \). Plugging in \( y'(0) = -1 \) gives \( -1 = \frac{1}{10} - 2k_1 - 3k_2 \). We solve this linear system to get \( k_1 = 11/5, k_2 = -11/10 \). So the solution to the IVP is

\[
y = \frac{1}{10} \sin(t) - \frac{1}{10} \cos(t) + \frac{11}{5} e^{-2t} - \frac{11}{10} e^{-3t}.
\]