MAP 2302, Exam II, Spring 2015

Name:________________________

Student signature:________________________

Write final answers on this sheet. Turn in all relevant work on separate sheets. Full work is required for full credit.

(1) [10] Use the method of undetermined coefficients to find the form of a particular solution y_p to the following ODEs. **Do not solve the equation!**

(a) [5] $y'' - 2y' + 5y = e^{-t}$
The characteristic polynomial is $r^2 - 2r + 5$ which has roots $r = 1 \pm 2i$. Following the method of undetermined coefficients we have $y_p = t^s A_0 e^{-t}$. Since -1 is not a root of the characteristic polynomial (equivalently e^{-t} is not a homogeneous solution) we choose $s = 0$ so that $y_p = A_0 e^{-t}$.

(b) [5] $y'' - 2y' + 5y = te^t \sin(2t)$
Following the same reasoning in this case we have

$$y_p = t\left((A_1 t + A_0)\, e^t \sin(2t) + (B_1 t + B_0)\, e^t \cos(2t)\right).$$

The factor of t at the beginning owes to the fact that $1 + 2i$ is a root of the characteristic polynomial.

(c) [5] $y'' - 2y' + 5y = e^{-t} + te^t \sin(2t)$
We already have a form to match $te^t \sin(2t)$ and e^{-t}. By the superposition principle we can add these forms (being careful not to resuse letters) to yield

$$y_p = t\left((A_1 t + A_0)\, e^t \sin(2t) + (B_1 t + B_0)\, e^t \cos(2t)\right) + C_0 e^{-t}.$$

(2) [10] Is it possible for $y_1 = e^t$ and $y_2 = t + 1$ to both be solutions to $y'' + p(t)y' + q(t)y = g(t)$ on $(-\infty, \infty)$ if p, q, g are all continuous on $(-\infty, \infty)$? **Justify your answer.** (**Hint:** Examine how the two functions intersect.)
No. These two functions satisfy the same initial conditions at $t = 0$, so if they were both solutions it would contradict the uniqueness theorem.

(3) [20] Find the general solution to the ODE

$$y'' - 4y' + 4y = te^t.$$

As always $y = y_p + y_h$. First we find y_h. The characteristic polynomial has a double root $r = 2$ so

$$y_h = k_1 e^{2t} + k_2 t e^{2t}.$$

To find y_p we will use the method of undetermined coefficients. The method gives us the form

$$y_p = (At + B)\, e^t$$

and so,

$$y_p' = (At + (A+B))\,e^t$$
$$y_p'' = ((At + (2A+B))e^t$$

Plugging this into the ODE gives

$$te^t\,[A - 4A + 4A] + e^t\,[2A + B - 4A - 4B + 4B] = te^t$$
$$Ate^t + (B - 2A)e^t = te^t$$

Equating the coefficients on the left and right side we get the linear system

$$A = 1$$
$$B - 2A = 0$$

which we can solve to obtain $A = 1, B = 2$ so

$$y = (t+2)e^t + k_1e^{2t} + k_2te^{2t}.$$

(4) [25] Given that t^2e^t and $(t^2+1)e^t$ are solutions to

$$ty'' + (1-2t)y' + (t-1)y = 4te^t, t > 0,$$

find the general solution to the ODE on $t > 0$.
Once more, the general solution is $y = y_p + y_h$. Since this is neither constant coefficient nor Cauchy-Euler, we need to develop a way to find y_h. Using the superposition principle, we find that $y_1 = (t^2+1)e^t - t^2e^t = e^t$ is one homogeneous solution. To find the y_2 we use the reduction of order formula. In standard form the ODE is

$$y'' + \frac{1-2t}{t}y' + \frac{t-1}{t}y = 4e^t.$$

Then the reduction of order formula becomes

$$y_2 = y_1 \int \frac{e^{-\int p(t)\,dt}}{(y_1)^2}\,dt$$
$$y_2 = e^t \int \frac{e^{-\int \frac{1-2t}{t}\,dt}}{e^{2t}}\,dt$$
$$y_2 = e^t \int \frac{e^{-\ln|t| + 2t}}{e^2t}\,dt$$

Since $t > 0$ we have $\ln|t| = \ln(t)$ and

$$y_2 = e^t \int \frac{t^{-1}e^{2t}}{e^{2t}}\,dt = e^t \int t^{-1}\,dt$$
$$y_2 = e^t \ln(t)$$

Finally, we may take $y_p = t^2e^t$ so the general solution is

$$y = t^2e^t + k_1e^t + k_2e^t\ln(t).$$

(5) [30] Solve the following Cauchy-Euler IVP:

$$t^2y'' - ty' + y = t,\ y(-1) = 1, y'(-1) = 2.$$

We still have $y = y_p + y_h$. We find y_h using the theory of Cauchy-Euler equations. The characteristic polynomial is $r^2 - 2r + 1$ which has a repeated root $r = 1$. We observe that the intial condition is given at $t = -1$ so we must use homogeneous solutions on $t < 0$. Then

$$y_h = k_1(-t) + k_2(-t)\ln(-t).$$

For definiteness, call $y_1 = -t, y_2 = -t\ln(-t)$. We will find y_p using reduction of order. In standard form the ODE is

$$y'' - \frac{1}{t}y' + \frac{1}{t^2}y = \frac{1}{t}.$$

The Wronskian

$$\begin{aligned} W = W[(-t), -t\ln(-t)] &= -t(-1 - \ln(-t)) - (-1)(-t\ln(-t)) \\ &= t + t\ln(-t) - t\ln(-t) \\ &= t. \end{aligned}$$

Then $y_p = v_1y_1 + v_2y_2$ where

$$\begin{aligned} v_1 &= \int \frac{-y_2g}{W}\,dt = \int \frac{t\ln(-t)t^{-1}}{t}\,dt \\ &= \int \frac{\ln(-t)}{t}\,dt \\ &= \frac{1}{2}\ln(-t)^2, \end{aligned}$$

and

$$\begin{aligned} v_2 &= \int \frac{y_1g}{W}\,dt = \int \frac{-tt^{-1}}{t}\,dt \\ &= -\int \frac{1}{t}\,dt = -\ln|t| \end{aligned}$$

Since $t < 0$ $\ln|t| = \ln(-t)$ and

$$v_2 = -\ln(-t).$$

Then

$$y_p = -\frac{t}{2}\ln(-t)^2 + t\ln(-t)^2 = \frac{t}{2}\ln(-t)^2.$$

and

$$\begin{aligned} y &= \frac{t}{2}\ln(-t)^2 + k_1(-t) + k_2(-t\ln(-t)) \\ y' &= \frac{1}{2}(2\ln(-t) + \ln(-t)^2) - k_1 + k_2(-1 - \ln(-t)) \end{aligned}$$

Plugging in $y(-1) = 1$ we get $-1 = k_1$. Similarly, $y'(-1) = 2$ Gives $2 = -k_1 - k_2$ so $k_2 = -1$. Then the solution to the IVP is

$$y = \frac{t}{2}\ln(-t)^2 + t + t\ln(-t).$$