MAP 2302, Exam III, Spring 2015

Name:____________________

Student signature:____________________

Write final answers on this sheet. Turn in all relevant work on separate sheets. Full work is required for full credit.

(1) Suppose that f, g are functions defined on $(-\infty, \infty)$. What is the convolution $f * g$? What if f, g are only defined on $[0, \infty)$?

This is just remembering definitions. In the former case we have

$$(f * g)(t) = \int_{-\infty}^{\infty} f(v)g(t-v)\,dv$$

and in the latter case we have

$$(f * g)(t) = \int_{0}^{t} f(v)g(t-v)\,dv$$

(2) If f is piecewise continuous and $|f(t)| \leq 10e^{5t}$, for what values of s is $F(s)$ guaranteed to exist?

f is piecewise continuous and of exponential order 5, so $F(s)$ exists for $s > 5$.

(3) Find the Laplace transform of the following functions

(a) $f(t) = t\sin(t)$

$$\begin{aligned} F(s) &= -\frac{d}{ds}\mathcal{L}\{\sin(t)\} \\ &= -\frac{d}{ds}\frac{1}{1+s^2} \\ &= \frac{2s}{1+s^2} \end{aligned}$$

(b) $f(t) = t^2$ if $0 < t < 2$ and f is periodic of period 2.

We have $f_2(t) = \Pi_{0,2}(t)t^2 = t^2 - u(t-2)t^2$ and so

$$\begin{aligned} F_2(s) &= \mathcal{L}\{t^2 - u(t-2)t^2\} = \mathcal{L}\{t^2\} - \mathcal{L}\{u(t-2)t^2\} \\ &= \frac{2}{s^3} - e^{-2s}\mathcal{L}\{(t+2)^2\} \end{aligned}$$

Since $(t+2)^2 = t^2 + 4t + 4$ we have

$$F_2(s) = \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right)$$

And so we find $F(s)$ by

$$F(s) = \frac{F_2(s)}{1-e^{-2s}} = \frac{1}{1-e^{-2s}}\left(\frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right)\right)$$

(c) $f(t) = \begin{cases} e^t & \text{if } 0 \le t < 5 \\ t & \text{if } t > 5 \end{cases}$

We rewrite $f(t) = \Pi_{0,5}(t)e^t + u(t-5)t = e^t - u(t-5)e^t + u(t-5)t$. Then

$$F(s) = \frac{1}{s-1} - e^{-5s}\mathcal{L}\{e^{t+5}\} + e^{-5s}\mathcal{L}\{t+5\}$$

Observe $e^{t+5} = e^5 e^t$ so that we have

$$F(s) = \frac{1}{s-1} - e^{-5s}\left(\frac{e^5}{s-1} + \frac{1}{s^2} + \frac{5}{s}\right)$$

(4) Use the Laplace transform to solve the following IVP

$$y'' - 2y' + y = 6t - 2;\ y(-1) = 3, y'(-1) = 7$$

We shift the initial conditions using the substitution $w(t) = y(t-1)$. With this the IVP becomes

$$w'' - 2w' + w = 6t - 8;\ w(0) = 3, w'(0) = 7$$

Taking the Laplace Transform we obtain

$$(s^2W - 3s - 7) - 2(sW - 3) + W) = \frac{6}{s^2} - \frac{8}{s} = \frac{6 - 8s}{s^2}$$

$$W(s^2 - 2s + 1) = \frac{6-8s}{s^2} + 3s + 1 = \frac{3s^3 + s^2 - 8s + 6}{s^2}$$

$$W = \frac{3s^3 + s^2 - 8s + 6}{s^2(s^2 - 2s + 1)}$$

$$W = \frac{3s^3 + s^2 - 8s + 6}{s^2(s-1)^2}$$

Now we do partial fractions to take $\mathcal{L}^{-1}$.

$$\frac{3s^3 + s^2 - 8s + 6}{s^2(s-1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2}$$

$$3s^3 + s^2 - 8s + 6 = As(s-1)^2 + B(s-1)^2 + C(s-1)s^2 + Ds^2$$

Equating coefficients of $s^3, s^2, s, 1$ on the left and right yields the equations

$$\begin{aligned} 3 &= A + C \\ 1 &= -2A + B - C + D \\ -8 &= A - 2B \\ 6 &= B \end{aligned}$$

Which can be solved to get $A = 4, B = 6, C = -1, D = 2$ so we have that

$$\frac{3s^3 + s^2 - 8s + 6}{s^2(s-1)^2} = \frac{4}{s} + \frac{6}{s^2} - \frac{1}{s-1} + \frac{2}{(s-1)^2}$$

Taking the inverse transform we get

$$w = 4 + 6t - e^t + 2te^t.$$

Doing back substitution yields

$$y(t) = w(t+1) = e + 6(t+1) - e^{t+1} + 2(t+1)e^{t+1}.$$

(5) Use the Laplace transform to solve the following IVP

$$ty'' - ty' + y = 2;\ y(0) = 2, y'(0) = -1$$

We take the Laplace transform using the rule $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)$. With this the original equation becomes

$$\begin{aligned} -\frac{d}{ds}(s^2Y - 2s + 1) + \frac{d}{ds}(sY - 2) + Y &= \frac{2}{s} \\ -s^2Y' - 2sY + 2 + sY' + Y + Y &= \frac{2}{s} \\ s(1-s)Y' + 2(1-s)Y &= \frac{2}{s} - 2 = \frac{2(1-s)}{s} \end{aligned}$$

Now, this is a first order equation which in standard form is

$$Y' + \frac{2}{s}Y = \frac{2}{s^2}$$

By Exam I material this has solution

$$Y = \frac{2}{s} + \frac{C}{s^2}$$

Taking the inverse transform we obtain that

$$y = 2 + Ct.$$

Seeing that $y'(0) = -1$, we choose $C = -1$ so $y = 2 - t$.

(6) Use the Laplace transform to solve the following symbolic IVP

$$y'' + 5y' + 6y = e^{-t}\delta(t-2);\ y(0) = 2, y'(0) = -5$$

Taking the Laplace transform we have

$$\begin{aligned} (s^2Y - 2s + 5) + 5(sY - 2) + 6Y &= e^{-2s}e^{-2} \\ Y(s^2 + 5s + 6) &= e^{-2s}e^{-2} + 2s + 5 \\ Y &= e^{-2}e^{-2s}\frac{1}{s^2 + 5s + 6} + \frac{2s + 5}{s^2 + 5s + 6} \end{aligned}$$

Here we do partial fractions (twice, but it's easy) to get

$$Y = e^{-2}e^{-2s}\left(\frac{1}{s+2} - \frac{1}{s+3}\right) + \left(\frac{1}{s+2} + \frac{1}{s+3}\right)$$

And so

$$y = e^{-2}u(t-2)\left(e^{-2(t-2)} - e^{-3(t-2)}\right) + \left(e^{-2t} + e^{-3t}\right)$$