MAP 2302, Exam III, Spring 2015

Name:________________________

Student signature:________________________

Write final answers on this sheet when able. Turn in all relevant work on separate sheets. Full work is required for full credit.

(1) [15] Find the Laplace transform of the following function:
$f(t) = 1 - (t-1)^2$ if $0 < t < 2$ and f is periodic of period 2.

$f_2(t) = \Pi_{0,2}(t)(1-(t-1)^2) = (u(t) - u(t-2))(2t - t^2)$ so we have

$$F_2(s) = \mathcal{L}\{(2t - t^2) - u(t-2)(2t-t^2)$$
$$= \left(\frac{2}{s^2} - \frac{2}{t^3}\right) - e^{-2s}\mathcal{L}\{2(t+2) - (t+2)^2\}$$

Since $2(t+2) - (t+2)^2 = -t^2 - 2t$ we have

$$F_2(s) = \left(\frac{2}{s^2} - \frac{2}{s^3}\right) - e^{-2s}\left(\frac{-2}{s^3} - \frac{2}{s^2}\right)$$

We conclude from this that

$$F(s) = \frac{\left(\frac{2}{s^2} - \frac{2}{s^3}\right) - e^{-2s}\left(\frac{-2}{s^3} - \frac{2}{s^2}\right)}{1 - e^{-2s}}$$

(2) [20] Find the inverse Laplace transform of the following functions:

(a) [12] $F(s) = \dfrac{2s-3}{s^2 - 4s + 8}$

Since $b^2 - 4ac = 16 - 40 = -36 < 0$ the denominator is irreducible. We will proceed by completing the square and trying to match the transforms of $e^{at}\sin(bt)$ and $e^{at}\cos(bt)$.

$$\frac{2s-3}{s^2-4s+8} = \frac{2s-3}{(s-2)^2+4}$$

We now see that $a = 2, b = 2$, so we rewrite the above as

$$\frac{2s-3}{(s-2)^2+4} = \frac{2(s-2)+1}{(s-2)^2+4}$$
$$= 2\frac{s-2}{(s-2)^2+4} + \frac{1}{2}\frac{2}{(s-2)^2+4}$$

Taking $\mathcal{L}^{-1}$ we obtain

$$f(t) = 2e^{2t}\cos(2t) + \frac{1}{2}e^{2t}\sin(2t)$$

(b) [8] $F(s) = e^{-2s}\dfrac{2s-3}{s^2-4s+8}$

From the previous result we can immediatley write

$$f(t) = u(t-2)(2e^{2(t-2)}\cos(2(t-2)) + \frac{1}{2}e^{2(t-2)}\sin(2(t-2)))$$

(3) [25] Solve the following IVP:

$$y'' + 3y' + 2y = \delta(t-2);\ y(1) = 0, y'(1) = 1$$

[+5] Is the solution continuous? Is its derivative continuous?

First we have to shift the initial conditions. Let $w(t) = y(t+1)$. The transformed IVP is

$$w'' + 3w' + 2w = \delta(t-1);\ w(0) = 0, w'(0) = 1$$

Taking the Laplace transform we obtain

$$\begin{aligned} s^2W - 1 + 3sW + 2W &= e^{-s} \\ W(s^2+3s+2) &= e^{-s} + 1 \\ W &= e^{-s}\frac{1}{s^2+3s+2} + \frac{1}{s^2+3s+2} \end{aligned}$$

The partial fractions work is omitted here, but one can obtain that

$$\frac{1}{s^2+3s+2} = \frac{1}{s+2} - \frac{1}{s+1}.$$

Then we can apply the inverse to get

$$w = u(t-1)\mathcal{L}^{-1}\left\{\frac{1}{s+2} - \frac{1}{s+1}\right\}(t-1) + \mathcal{L}^{-1}\left\{\frac{1}{s+2} - \frac{1}{s+1}\right\}(t)$$

Since $\mathcal{L}^{-1}\left\{\frac{1}{s+2} - \frac{1}{s+1}\right\}(t) = e^{-2t} - e^{-t}$ we have

$$w = u(t-1)\left(e^{-2(t-1)} - e^{-(t-1)}\right) + \left(e^{-2t} - e^{-t}\right)$$

Finally we use $w(t-1) = y(t)$ to obtain the solution

$$y(t) = u(t-2)\left(e^{-2(t-2)} - e^{-(t-2)}\right) + \left(e^{-2(t-1)} - e^{-(t-1)}\right)$$

(4) Answer the following related questions:

(a) [15] Take the Laplace transform of the following IVP:

$$t^2y'' + 4ty' + 2y = t + 2;\ y(0) = 2, y'(0) = 1$$

$$\begin{aligned} \mathcal{L}\{t^2y''\} &= \frac{d^2}{ds^2}\mathcal{L}\{y''\}(s) \\ &= \frac{d^2}{ds^2}\left(s^2Y - sy(0) - y'(0)\right) \\ &= \frac{d}{ds}\left(s^2Y' + 2sY - y(0)\right) \\ &= s^2Y'' + 2sY' + 2sY' + 2Y \\ &= s^2Y'' + 4sY' + 2Y \end{aligned}$$

By the same process we get $\mathcal{L}\{ty'\} = -(sY' + Y)$. Then the transform of the above is

$$\begin{aligned} s^2Y'' + 4sY' + 2Y - 4sY' - 4Y + 2Y &= \frac{1}{s^2} + \frac{2}{s} \\ s^2Y'' &= \frac{1}{s^2} + \frac{2}{s} \\ Y'' &= \frac{1}{s^4} + \frac{2}{s^3} \end{aligned}$$

(b) [10] Find y if $\frac{d^2}{ds^2}Y(s) = \frac{1}{s^4} + \frac{2}{s^3}$ Since $\frac{d^2}{ds^2}Y(s) = \mathcal{L}\{t^2y\}$ we can rewrite the equation as

$$\mathcal{L}\{t^2y\} = \frac{1}{s^4} + \frac{2}{s^3}$$

Taking $\mathcal{L}^{-1}$ gives

$$\begin{aligned} t^2y &= \frac{t^3}{6} + t^2 \\ y &= \frac{t}{6} + 1 \end{aligned}$$

(c) **[+5]** Why isn't y from part (b) a solution to the IVP from part (a)?

(5) [5] Write a formula for computing $\mathcal{L}\{y'''\}$ in terms of $\mathcal{L}\{y\}$.

This one is straight from the table. Apply the formula for $\mathcal{L}\{y'\}$ three times and you get

$$\mathcal{L}\{y'''\} = s^3Y - s^2y(0) - sy'(0) - y''(0)$$

(6) [5] Suppose that f is continuous and periodic. For what values of s is $F(s)$ guaranteed to exist? **Justify your answer.**

If f is of period T, then so is $|f(t)|$. Then $|f|$ has a maximum value M on $[0, T]$ and hence M is a maximum value on $[0, \infty)$. Then $|f(t)| \leq M = Me^{0t}$ gives that f is of exponential order 0 so $F(s)$ exists for $s > 0$.

(7) [5] Given a function f which is PWC and of exponential order find $(f * \delta)(t)$. **Hint:** Either use the definition of convolution or use $\mathcal{L}$.

$(f * \delta)(t) = \int_{-\infty}^{\infty} f(v)\delta(t - v)dv = f(t)$. Or by taking $\mathcal{L}$ we have $\mathcal{L}\{f * \delta\} = \mathcal{L}\{f\} * \mathcal{L}\{\delta\} = \mathcal{L}\{f\}$. Taking the inverse transform gives $(f * \delta)(t) = f(t)$

(8) [**+5**] Find a continuous function f defined on $[0, \infty)$ which is not of exponential order α for any choice of α but $\mathcal{L}\{f\}(s)$ exists for $s > 0$.