MAP 2302, Exam IV, Spring 2015

Name:________________________

Student signature:________________________

Turn in all relevant work with final answers circled on separate sheets. Full work is required for full credit.

(1) [25] Solve the following symbolic IVP

$$y'' + 2y' + 2y = \delta(t - 2\pi)\,;\, y(\pi) = 2, y'(\pi) = 1$$

First, shift the initial conditions by setting $w(t) = y(t+\pi)$. Then the shifted IVP (don't forget to shift the right hand side too) becomes

$$w'' + 2w' + 2w = \delta(t - \pi)\,;\, w(0) = 2, w'(0) = 1.$$

Taking $\mathcal{L}$ we get

$$s^2W - 2s - 1 + 2(sW - 2) + 2W = e^{-\pi s}$$
$$W(s^2 + 2s + 2) = e^{-\pi s} + 2s + 5$$

And so we obtain that

$$W = e^{-\pi s}\frac{1}{s^2 + 2s + 2} + \frac{2s + 5}{s^2 + 2s + 2}$$

The denominator is not factorable over the real numbers ($2^2 - 4 * 2 < 0$), so our goal is to complete the square in the denominator and match the form of the transforms of $e^{at}\sin(bt)$ and $e^{at}\cos(bt)$.

$$W = e^{-\pi s}\frac{1}{(s+1)^2 + 1} + 2\frac{(s+1)}{(s+2)^2 + 1} + 3\frac{1}{(s+1)^2 + 1}$$

Taking $\mathcal{L}^{-1}$ we have

$$w = u(t - \pi)e^{-(t-\pi)}\sin(t - \pi) + 2e^{-t}\cos(t) + 3e^{-t}\sin(t)$$

Shifting back by $y(t) = w(t - \pi)$ we get

$$y = u(t - 2\pi)e^{-(t-2\pi)}\sin(t - 2\pi) + 2e^{-(t-\pi)}\cos(t - \pi) + 3e^{-(t-\pi)}\sin(t - \pi)$$

Or,

$$y = u(t - 2\pi)e^{-(t-2\pi)}\sin(t) - 2e^{-(t-\pi)}\cos(t) - 3e^{-(t-\pi)}\sin(t)$$

(2) [25] Answer the following questions about the ODE

$$y' + 4xy = 0$$

- [15] Find a recurrence relation that determines the coefficients a_n of a power series for the solution y.

Set $y = \sum_{n=0}^{\infty} a_n x^n$ so $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$. Then the ODE becomes

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 4x \sum_{n=0}^{\infty} a_n x^n = 0$$
$$\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 4 a_n x^{n+1} = 0$$

Normalizing the exponent of x we get

$$\sum_{k=0}^{\infty} (k+1) a_{k+1} x^k + \sum_{k=1}^{\infty} 4 a_{k-1} x^k = 0$$
$$a_1 + \sum_{k=1}^{\infty} \left[(k+1)\, a_{k+1} + 4a(k-1) \right] x^k = 0$$

Equating coefficients of powers of x on both sides yields the recurrence relation

$$a_1 = 0$$
$$a_{k+1} = \frac{-4 a_{k-1}}{k+1}, k \geq 1$$

- [5] What is the radius of convergence for that power series?

  Since $4x$ is analytic everywhere, $R = \infty$.
- [5] Use the recurrence relation to find an explicit formula (closed form) for a_n.

  Note that since $a_1 = 0$ for every odd n $a_n = 0$. For even n we keep multiplying by (-4) and dividing by the "next" even number. For example

$$a_0 = a_0$$
$$a_2 = a_0 \frac{-4}{2}$$
$$a_4 = a_2 \frac{-4}{4} = a_0 \frac{(-4)^2}{2*4}$$
$$a_6 = a_4 \frac{-6}{6} = a_0 \frac{(-4)^3}{2*4*6}$$

  We conclude $a_{2n} = a_0 \frac{(-4)^n}{2*4*\cdots*(2n)} = \frac{(-2)^n}{n!}$.
- **[+5]** Find an explicit (non-series) formula for y.

  Either compare to the series for e^x or use Exam I material to see $y = e^{-2x^2}$.

(3) [25] Find a recurrence relation for the coefficients of a power series for a general solution to

$$(x^2+1)y''+y=0$$

centered around $x=1$. Use the recurrence relation ([15]) to find the first four nonzero terms ([5]) of the series. What is the minimum radius of convergence ([5]) of the series?

Shift by taking $w(x)=y(x+1)$. Then the shifted ODE becomes

$$((x+1)^2+1)w''+w=0$$
$$(x^2+2x+2)w''+w=0$$

Set $w=\sum_{n=0}^{\infty}a_nx^n$, $w''=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}$. Then after multiplying the powers of x through the ODE becomes

$$\sum_{n=2}^{\infty}n(n-1)a_nx^n+\sum_{n=2}^{\infty}2n(n-1)a_nx^{n-1}+\sum_{n=2}^{\infty}2n(n-1)a_nx^{n-2}+\sum_{n=0}^{\infty}a_nx^n=0$$

Normalizing the powers of x we get

$$\sum_{k=2}^{\infty}k(k-1)a_kx^k+\sum_{k=1}^{\infty}2k(k+1)a_{k+1}x^k+\sum_{k=0}^{\infty}2(k+2)(k+1)a_{k+2}x^k+\sum_{k=0}^{\infty}a_kx^k=0$$

We remove the $k=0,1$ terms and combine sums to obtain

$$4a_2+a_0+(4a_2+12a_3+a_1)x+$$
$$+\sum_{k=2}^{\infty}\left[(k^2-k+1)a_k+2k(k+1)a_{k+1}+2(k+2)(k+1)a_{k+2}\right]x^k=0$$

This yields the recurrence relation

$$a_2=\frac{-1}{4}a_0$$
$$a_3=-\frac{1}{3}a_2-\frac{a_1}{12}=\frac{1}{12}a_0-\frac{a_1}{12}$$
$$a_{k+2}=\frac{-(k^2-k+1)a_k-2k(k+1)a_{k+1}}{2(k+2)(k+1)},k\geq 2$$

Thankfully we only need a_0 to a_3 to get the first four terms. Looking above we have

$$w=a_0+a_1x-\frac{1}{4}a_0x^2+\frac{1}{12}\left(a_0-a_1\right)x^3$$

And since $y(x)=w(x-1)$

$$y=a_0+a_1(x-1)-\frac{1}{4}a_0(x-1)^2+\frac{1}{12}\left(a_0-a_1\right)(x-1)^3$$

The singular points occur then $(1+x^2)=0$. So $x=\pm i$. The distinace from 1 to $\pm i$ is $\sqrt{2}$ so the minimum radius of convergence is $R=\sqrt{2}$.

(4) [25] Find the first four nonzero terms of a power series solution centered at $x = 0$ to the IVP

$$y'' - \sin(x)y = \cos(x)\ ; y(0) = 1, y'(0) = 1.$$

Recall that

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$a_0 = y(0) = 1$ and $a_1 = y'(0) = 1$. We can use the "bootstrapping" method to find a_3, a_4. We have

$$y'' = \cos(x) + \sin(x)y$$

Differentiating yields

$$y''' = -\sin(x) + \sin(x)y' + \cos(x)y$$

Plugging in 0 into these gives $y''(0) = 1$ and $y'''(0) = y(0) = 1$. Since $a_n = \frac{y^{(n)}}{n!}$ our approximation is

$$y = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$$

(5) [**+5**] Use the root test to show that the power series $\Sigma_{n=1}^{\infty} n a_n x^{n-1}$ has the same radius of convergence as the power series for $\Sigma_{n=0}^{\infty} a_n x^n$. In other words, the series for y' has the same radius of convergence as the series for y.

(6) [**+5**] Give an example of a function f for which the nth derivative $f^{(n)}$ is continuous for all n but f is not analytic at 0. For full credit, provide justification.

$f(t)$	$\mathcal{L}\{f\}(s)$
1	$\frac{1}{s}$
$t^n, n = 0, 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$
$\sin(bt)$	$\frac{b}{s^2 + b^2}$
$\cos(bt)$	$\frac{s}{s^2 + b^2}$
$u(t-a)f(t-a)$	$e^{-as}\mathcal{L}\{f(t)\}(s)$
$\delta(t-a)$	e^{-as}
$e^{at}f(t)$	$\mathcal{L}\{f(t)\}(s-a)$
$f'(t)$	$s\mathcal{L}\{f(t)\}(s) - f(0)$
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(t)\}(s)$
$f(t)$ (period T)	$\frac{\mathcal{L}\{f_T(t)\}}{1 - e^{-Ts}}$
$(g * h)(t)$	$\mathcal{L}\{g(t)\}(s)\mathcal{L}\{h(t)\}(s)$