Contents

MAA6406-07 COURSE NOTES 2016-17

1. Integral formulas

Theorem 1.1 (Bounded Convergence)**.** *Suppose $f, f_n : [a,b] \to \mathbb{C}$. If*

(i) f, f_n are continuous;
(ii) (f_n) converges to f pointwise;
(iii) there is an M such that $|f_n| \leqslant M$ (uniformly),

then $\int f_n$ converges to $\int f$.

Remark 1.2. Uniform, rather than pointwise convergence, implies the uniform boundedness conditions. With this stronger hypothesis, the proof of the Theorem is very straightforward. ◇

Theorem 1.3 (Leibniz Rule)**.** *If $\varphi : [a,b] \times [c,d] \to \mathbb{C}$ and φ_2, the partial derivative with respect to the second variable, exists and is continuous, then $g : [c,d] \to \mathbb{C}$ defined by*

$$g(t) = \int_a^b \varphi(s,t)\, ds$$

is continuously differentiable and

$$g'(t) = \int_a^b \varphi_2(s,t)\, ds.$$

Proof. It suffices to prove the theorem in the case the codomain of φ is $\mathbb{R}$. In this case the codomain of g is also $\mathbb{R}$. Since φ_2 is continuous, there is an M such that $|\varphi_2(s,t)| \leqslant M$ for all s, t. Fix $t_0 \in [c,d]$ and suppose (t_n) is a sequence from $[c,d]$ that converges to t_0 (with $t_n \neq t_0$). Define $f_n : [a,b] \to \mathbb{R}$ by

$$f_n(s) = \frac{\varphi(s,t_n) - \varphi(s,t_0)}{t_n - t_0}.$$

By the MVT, for each s and n, there exists a point c between t_n and t_0 such that

$$|f_n(s)| = |\varphi_2(s,c)| \leqslant M.$$

Hence the sequence (f_n) is uniformly bounded. By the differentiability hypothesis on φ, the sequence $f_n(s)$ converges pointwise to $f_0(s) = \varphi(s,t_0)$. The result now follows from the bounded convergence theorem, Theorem 1.1. □

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Theorem 1.4 (An M test)**.** *Suppose (g_n) is a uniformly bounded sequence series of continuous $\mathbb{C}$-valued functions defined on an interval $[a,b]$. If (a_n) is a sequence from $\mathbb{C}$ and if $\sum a_n$ converges absolutely, then the series $\sum a_n g_n(s)$ converges to a continuous $g:[a,b]\to\mathbb{C}$ and moreover,*

$$\int_a^b g ds = \sum_{n=1}^{\infty} \int_a^b g_n ds = \lim_{N\to\infty} \sum_{n=1}^{N} \int_a^b g_n ds.$$

Lemma 1.5. *If $|z|<1$, then $\int_0^{2\pi} \frac{\exp(is)}{e^{is}-z} ds = 2\pi$. (If $|z|>1$, then the integral is 0).* †

Proof. In the case $|z|<1$, apply the bounded convergence theorem to

$$\frac{1}{1-e^{-is}z} = \sum (e^{-is}z)^n;$$

i.e., $g_n(s) = e^{-ins}$ and $a_n = z^n$. The case $|z|>1$ is similar. □

Theorem 1.6 (Cauchy Integral Version 0)**.** *Suppose $f:G\to\mathbb{C}$ is analytic, $y\in\mathbb{C}$ and $r>0$. If $\overline{B(y;r)}\subset G$ and $\gamma:[0,2\pi]\to G$ is defined by $\gamma(s)=y+re^{is}$, then, for $|w-y|<r$,*

$$f(w) = \frac{1}{2i\pi}\int_\gamma \frac{f(z)}{z-w} dz. \tag{1}$$

Alternately, letting $F_w(z) = \frac{f(z)}{z-w}$,

$$f(w) = \int_\gamma F_w = \int F_w \circ \gamma \, d\gamma.$$

On the other hand, if $|w-y|>r$, then the integral on the right hand side of equation (1) is 0.

Remark 1.7. The Lemma recovers the values f inside the circle traversed by γ from the values of f on γ. In particular, $f(y)$ is the average value of f on the circle γ. ◇

Proof. Suppose, without loss of generality, $y=0$ and $r=1$ and let $|w|<1$ be given. It suffices to prove,

$$0 = \int_0^{2\pi} \left[f(w) - \frac{f(e^{is})}{e^{is}-w}\, e^{is} \right] ds.$$

Note, by convexity, for $0\leqslant t\leqslant 1$, that $|w+t(e^{is}-w)| = |(1-t)w+te^{is}| \leqslant 1$ and hence we may define $\varphi:[0,2\pi]\times[0,1]\to\mathbb{C}$ by

$$\varphi(s,t) = \frac{f(w+t(e^{is}-w))}{e^{is}-w}\, e^{is} - f(w).$$

Define $g:[0,1]\to\mathbb{C}$ by

$$g(t) = \int_0^{2\pi} \varphi(s,t)\, ds.$$

It suffices to show $g(1) = 0$. By Theorem 1.3,

$$g'(t) = \int_0^{2\pi} \varphi_2(s,t)\,ds.$$

Now

$$\varphi_2(s,t) = f'(w + t(e^{is} - w))e^{is} =: G_t(s).$$

On the other hand, $G_t(s) = \Phi_t'(s)$ where

$$\Phi_t(s) = \frac{1}{it} f(w + t(e^{is} - w)),$$

for $0 < t \leqslant 1$. Hence, for $0 < t \leqslant 1$, by the FTC $g'(t) = \Phi_t(2\pi) - \Phi_t(0) = 0$. Hence g is constant on $(0, 2\pi]$ and thus, by continuity on all of $[0, 2\pi]$. Hence $g(1) = g(0)$. On the other hand,

$$g(0) = \int_0^{2\pi} \left[\frac{f(w)}{e^{is} - w} - f(w)\right] ds = f(w) \int_0^{2\pi} \left[\frac{1}{e^{is} - w} - 1\right] ds.$$

The right hand side is 0 by Lemma 1.5. Hence $g(1) = 0$ as desired. □

2. Power Series Representations

Theorem 2.1 (Power Series Representation, Part I)**.** *Let $\Omega \subset \mathbb{C}$ be an open set. If $f : \Omega \to \mathbb{C}$, $y \in \Omega$ and $r > 0$ and $\overline{B(y;r)} \subset \Omega$, then there is an M (depending on r) such that the sequence*

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(y + re^{is})e^{-ins}}{r^n}\,ds$$

satisfies $|a_n| \leqslant \frac{M}{r^n}$ and hence the series

$$\sum_{n=0}^{\infty} a_n w^n$$

has radius of convergence r and for $|z - a| < r$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - y)^n.$$

In particular, the a_n do not depend upon r.

Proof. Suppose, without loss of generality $y = 0$. In this case, if $|z| < r$, then by Theorem 1.6,

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z}\,dz$$

where $\gamma : [0, 2\pi] \to G$ is defined by $\gamma(s) = re^{is}$. On the other hand,

$$\int_\gamma \frac{f(w)}{w - z}\,dz = \int_0^{2\pi} \frac{f(e^{is})}{1 - \frac{z}{re^{-is}}} ds.$$

The sequence $g_n(s) = f(re^{is})e^{-ins}$ is uniformly bounded (on the interval $[0, 2\pi]$). An application of Theorem 1.4 with $a_n = (\frac{z}{r})^n$ gives

$$\int_0^{2\pi} \frac{f(re^{is})}{1 - \frac{z}{re^{-is}}}\, ds = i \lim_{N\to\infty} \sum_{n=1}^{N} \int_0^{2\pi} f(re^{is})e^{-ins}(\frac{z}{r})^n\, ds$$
$$= i \sum_{n=0}^{\infty} \left[\int_0^{2\pi} \frac{f(re^{is})e^{-ins}}{r^n}\, ds \right] z^n.$$

□

Corollary 2.2 (Power Series Representations, Part II)**.** *Suppose $\Omega \subset \mathbb{C}$ is an open set, $f : \Omega \to \mathbb{C}$ is analytic, $y \in \Omega$, $r > 0$ and $B(y; r) \subset \Omega$.*

(i) Letting R denote the distance from y to $\partial\Omega$, the function f has a power series representation on $B(y; R)$;

$$f(z) = \sum_{n=0}^{\infty} a_n(z-y)^n,\ |z-y| < R$$

(ii) f is infinitely differentiable, and $f^{(n)}(y) = n!a_n$;

(iii) further,

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-y)^{n+1}}\, dw,$$

where $\gamma : [0, 2\pi] \to \Omega$ is the path $\gamma(s) = y + re^{is}$;

(iv) (Cauchy estimate) if $|f|$ is uniformly bounded by M in $B(y; r)$, then

$$|f^n(y)| \leqslant \frac{n!M}{r^n};$$

(v) there exists an analytic function $F : B(y; r) \to \mathbb{C}$ such that $f|_{B(y;r)} = F'$; i.e., F has a primitive on $B(y; r)$;

(vi) if $\gamma : [a, b] \to B(y; r)$ is a rectifiable curve, then

$$\int_\gamma f = 0.$$

Proof. For item (v): $f = \sum a_n(z-y)^n$ has radius of convergence $r > 0$. Hence so does the series $\sum \frac{a_n}{n+1}(z-y)^{n+1}$ and thus

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-y)^{n+1}$$

defines an analytic function on $B(y; r)$ whose derivative is f. Item (vi) follows from item (v) by the fundamental theorem of line integrals, Corollary 1.22 in Conway. □

Recall a *domain* is a an open connected set $G \subset \mathbb{C}$.

Proposition 2.3. *Suppose $G \subset \mathbb{C}$ is a domain. For an analytic function $f : G \to \mathbb{C}$, the following are equivalent.*

(i) there is an open set $U \subset G$ such that $f(z) = 0$ for all $z \in U$;
(ii) there is a point $y \in G$ such that $f^{(n)}(y) = 0$ for all natural numbers n;
(iii) $f^{(n)}(y) = 0$ for every point $y \in G$ and every natural number n;
(iv) $f \equiv 0$; i.e., f is the zero function.

Remark 2.4. *If natural numbers n are replaced with positive integers n, then, choosing any point $p \in G$ and replacing f by $f - f(p)$, the conclusion of item (iv) becomes f is constant.*

Proof. Let Z denote the set $\{w \in G : f^{(n)}(w) = 0 \text{ for all natural numbers } n\}$. Given $w \in Z$, consider the power series expansion of f about w, valid for $|z - w| < r_w$, where r_w is the distance from w to the boundary of G,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - w)^n.$$

Since $w \in Z$, all the $a_n = 0$ and hence f vanishes on the open set $\{|z-w| < r\}$. Reversing the process, if f vanishes on the open set $U \subset G$, then $U \subset Z$. Hence Z is open and it contains all open sets on which f vanishes identically.

The equivalence of item (i) and item (iii), the equivalences of items (iii) and item (iv) and the implication item (iii) implies item (ii) are all now evident. Thus, to complete the proof, it suffices to prove item (ii) implies item (iii). If, for some $y \in G$, all the derivative of f at y are 0, then Z is not empty. On the other hand, it is a closed set (subset of G), being the countable intersection of the sets,

$$Z_n = g_n^{-1}(\{0\}),$$

where g_n is the continuous function $f^{(n)}$. It follows, by connectedness of G, that $G = Z$. □

Theorem 2.5 (Maximum Modulus)**.** *Suppose $G \subset \mathbb{C}$ is a domain. If $f : G \to \mathbb{C}$ is analytic and not constant, then the function $|f| : G \to \mathbb{R}_{\geqslant 0}$ does not have a maximum on G.*

Proof. Suppose $f : G \to \mathbb{C}$ and there is a point $y \in G$ such that $|f(y)| \geqslant |f(z)|$ for all $z \in G$ and we may assume $|f(y)| > 0$. There is an $R > 0$ such that $B(y; R) \subset G$. Fix $0 < r < R$ and letting γ denote the curve, $\gamma(s) = y + re^{is}$, $s \in [0, 2\pi]$, observe by Remark 1.7,

$$f(y) = \frac{1}{2i\pi} \int_0^{2\pi} f(y + re^{is})\, ds.$$

Hence,

$$|f(y)| \leqslant \frac{1}{2\pi} \int_0^{2\pi} |f(y + re^{is})|\, ds \leqslant |f(y)|.$$

Hence, $|f(y + re^{is})| = |f(y)|$ for all $s \in \mathbb{R}$ and $0 \leqslant r < R$. It follows that $|f(z)| = |f(y)|$ on $B(y; R)$. If $f(y) = 0$, the proof is complete. Otherwise, the range of f lies in the circle centered to 0 of radius $|f(y)|$ and therefore, by Problem 2.1, f is constant. □

2.1. **Problems.**

Problem 2.1. Show, if $G \subset \mathbb{C}$ is a domain and $f : G \to \mathbb{C}$ is real-valued, then f is constant. (Suggestion: use the Cauchy Riemann equations.)

Show if the range of f lies in a circle C, then f is constant. (Suggestion: Given a point $y \in G$, choose a neighborhood $y \in U \subset G$ such that $f(U)$ is not all of C (and be sure to explain why this choice is possible), and then compose with a map taking $f(U)$ into the real line.

Suppose $f, g : G \to \mathbb{C}$. Show, if $|f| = |g|$, then there is a $c \in \mathbb{C}$ such that $g = cf$.

3. The theorems of Morera and Goursat

Given an open set $\Omega \subset \mathbb{C}$ and, a function $f : \Omega \to \mathbb{C}$ and points $x, y \in \Omega$ such that the *line segment* $[\![x, y]\!] = \{x + t(y - x) : 0 \leqslant t \leqslant 1\} \subset \Omega$, let

$$\int_x^y f = (y - x) \int_0^1 f(x + s(y - x))\, ds.$$

Hence this integral is the line integral over the path $\gamma : [0, 1] \to \Omega$ defined by $\gamma(s) = x + s(y - x)$. Given points $x, y, w \in \Omega$ let $T_{x,y,w}$ denote the closed path $[\![x, y, w, x]\!] := [\![x, y]\!] + [\![y, w]\!] + [\![w, x]\!]$ (assuming of course all these segments lie in Ω). We call T an *oriented triangle*. We also let T denote the triangle T together with its interior (which is intended will be clear from the context) and say T lies in Ω if $T \subset \Omega$.

Theorem 3.1 (Morera)**.** *Suppose $\Omega \subset \mathbb{C}$ is open. If $f : \Omega \to \mathbb{C}$ is continuous and if $\int_T f = 0$ for all oriented triangles in Ω, then f is analytic. In fact, if $y \in \Omega$, $r > 0$ and $B(y; r) \subset \Omega$, then the function $F : B(y; r) \to \mathbb{C}$ defined by*

$$F(z) = \int_x^z f \tag{2}$$

is differentiable and $F' = f|_{B(x;r)}$; i.e, locally f has primitive.

Proof. It suffices to prove the function F of equation (2) is differentiable and $F'(z) = f(z)$ for $z \in B(y; r)$. Accordingly, fix $y \in \Omega$ and $r > 0$ such that $B(y; r) \subset \Omega$. Define, for $z \in B(y; r)$,

$$F(z) = \int_y^z f.$$

Given $z, w \in B(y; r)$, consider the triangle $T = [\![y, z, w, y]\!]$. By hypothesis,

$$F(z) - F(w) = \int_w^z f.$$

Thus,

$$\frac{F(z) - F(w)}{z - w} - f(w) = \int_0^1 [f(w + s(z - w)) - f(w)]\, ds$$

and therefore

$$\left|\frac{F(z)-F(w)}{z-w} - f(w)\right| \leqslant \int_0^1 |f(w+s(z-w))-f(w)|\, ds.$$

An appeal to the continuity of f shows F is differentiable and $F'(w) = f(w)$. Since $F' = f$ and f is continuous, F is analytic. Hence f is analytic on $B(y;r)$ and the proof is complete. □

Theorem 3.2 (Goursat)**.** *If $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$ is differentiable, then f is analytic; i.e., f' is continuous.*

Sketch of proof. By Theorem 3.1, it suffices to prove $\int_T f = 0$ for every oriented triangle T lying in Ω. Accordingly, let an oriented triangle T lying in Ω be given. Divide T into four triangles $T_1, \dots, T_4$ using the midpoints of the sides of T and oriented so that

$$\int_T f = \sum_{j=1}^{4} \int_{T_j} f.$$

Choose $1 \leqslant k \leqslant 4$ such that, for each $1 \leqslant j \leqslant 4$,

$$|\int_{T_k} f| \geqslant |\int_{T_j} f|.$$

Letting $T^{(1)} = T_k$,

$$|\int_T f| \leqslant 4|\int_{T^{(1)}} f|.$$

Continuing in this fashion, construct a nested decreasing sequence of oriented triangles $T^{(m)}$ such that

(1) the length L_m of the boundary of $T^{(m)}$ is $2^{-m}L$, where L is the length of the boundary of T;

(2)
$$|\int_T f| \leqslant 4^m|\int_{T^{(m)}} f|$$

(3) the diameter D_m of the $T^{(m)}$ is $2^{-m}D$, where D is the diameter of T.

Since also each $T^{(m)}$ is closed, there exists a unique point $p \in \cap_m T^{(m)}$. Using differentiability of f at p, given $\epsilon > 0$, choose δ such that, for $|z-p| < \delta$,

$$|f(z) - f(p) - f'(p)(z-p)| < \epsilon|z-p|.$$

A direct calculation shows, for any triangle S,

$$\int_S z = \int_S 1 = 0.$$

Thus, for m sufficiently large,

$$\begin{aligned}|\int_{T^{(m)}} f| =& \left|\int_{T^{(m)}} [f(z) - f(p) - f'(p)(z-p)]\right| \\ \leqslant& \epsilon D_m L_m = \epsilon 4^{-m}.\end{aligned}$$

It follows that

$$|\int_T f| \leqslant \epsilon$$

and the proof is complete. □

4. Zeros of analytic functions and the fundamental theorem of algebra

An *entire function* is an analytic function $f : \mathbb{C} \to \mathbb{C}$.

Proposition 4.1. *If f is entire, then f has a power series representation at the origin with infinite radius of convergence.*

The power series representation, with infinite radius of convergence, can actually be taken about any point $y \in \mathbb{C}$.

Theorem 4.2 (Louiville)**.** *If f is entire and bounded, then f is constant.*

Proof. Write the power series representation of f as

$$f = \sum_{n=0}^{\infty} a_n z^n.$$

By hypothesis, there is an $M \in \mathbb{R}_{>0}$ such that $|f(z)| \leqslant M$ uniformly in z. By the Cauchy estimate, Corollary 2.2(iv), for every $R \in \mathbb{R}_{>0}$,

$$|f^{(n)}(0)| \leqslant n! \frac{M}{R^n}.$$

Hence $f^{(n)}(0)$ is zero for each positive integer n. The conclusion now follows from Remark 2.4. □

Given a set X, a *zero of a function* $f : X \to \mathbb{C}$ is a point $y \in X$ such that $f(y) = 0$. A polynomial p is an entire function of the form, $p(z) = \sum_{j=1}^{d} p_j z^j$. It has *degree* d in the case $p_d \neq 0$. It is *monic* if $p_d = 1$. Note that p is constant if and only if $p_j = 0$ for $j > 0$.

Theorem 4.3 (Fundamental Theorem of Algebra)**.** *A non-constant polynomial $p : \mathbb{C} \to \mathbb{C}$ has a zero.*

If p is a monic polynomial of degree d, then there exists a $1 \leqslant k \leqslant d$, distinct $a_1, \ldots, a_k \in \mathbb{C}$ and positive integers n_j such that

$$p(z) = \prod_{j=1}^{k} (z - a_j)^{n_j}$$

and $\sum_{j=1}^{k} n_j = d$.

Lemma 4.4. *If p is a non-constant polynomial, then for each $C \in \mathbb{R}_{>0}$ there exists an $R \in \mathbb{R}_{>0}$ such that $|p(z)| \geqslant C$ for all $|z| > R$; i.e., $\lim_{|z| \to \infty} p(z) = \infty$.*

Proof of Theorem 4.3. Let p be a polynomial with no zeros. In this case $f = p^{-1}$ is entire. By Lemma 4.4, there is an $R \in \mathbb{R}_{>0}$ such that $|f(z)| \leqslant 1$ for $|z| > R$. On $\overline{B(0;R)}$ the function f is continuous and hence (uniformly) bounded. Thus f is a bounded entire function and hence, by Louivilles Theorem 4.2, constant. Thus p is constant.

We prove the second part by induction on the degree d of p. It is evidently true for a monic polynomial of degree $d = 1$. Suppose the result is true for all polynomials of degree d and p is a monic polynomial of degree $d+1$. By the first part of the theorem, p has a zero y. By a very special case of the Euclidean algorithm (we do need a little something from algebra) there exists a polynomials s and r of degree $d+1-1 = d$ and at most $1-1 = 0$ respectively such that $p(z) = (z-a)s(z)+r(z)$. (Equivalently, write p in terms of powers of $(z-y)^j$ and observe $p(y) = 0$ implies $p(z) = \sum_{j=1}^{d} c_j(z-y)^j$.) In particular r is constant and from $p(y) = 0$ it follows that $r = 0$. Hence $p = (z-y)s$, where s has degree d. An application of the induction hypothesis completes this induction step. □

A subset D of a domain $G \subset \mathbb{C}$ is a *determining set* (for G) if the only analytic function $f : G \to \mathbb{C}$ that is zero on D is the 0 function. It immediately follows that any two functions that agree on D are the same.

Proposition 4.5. *Let $G \subset \mathbb{C}$ be a domain. If $f : G \to \mathbb{C}$ is analytic, $y \in G$ and $f(y) = 0$, then either f is identically 0, or there is an n and an analytic function $g : G \to \mathbb{C}$ such that $g(y) \neq 0$ and $f(z) = (z-y)^n g(z)$. Moreover, n is characterized by the property $f^{(m)}(y) = 0$ for $m \leqslant n-1$ and $f^{(n)}(y) \neq 0$.*

Proof. If f is not identically zero, but $f(y) = 0$, then there is a positive integer n such that $f^{(m)}(y) = 0$ for $m \leqslant n-1$ and $f^{(n)}(y) \neq 0$ by Proposition 2.3 (ii). Thus, there is an $r > 0$ such that f has the power a power series expansion,

$$f(z) = \sum_{j=n}^{\infty} \frac{f^{(j)}(y)}{n!}(z-y)^j = (z-a)^n \sum_{j=0}^{\infty} a_{n+j}(z-y)^j,$$

on $B(y;r)$. Thus, there is an analytic function h defined on $B(y;r)$ such that $f(z) = (z-y)^n h(z)$ on $B(y;r)$. Define $g : G \to \mathbb{C}$ by $g(z) = \frac{f(z)}{(z-y)^n}$ for $z \neq y$ and $h(z)$ for $z \in B(y;r)$. □

The *order of a zero* y of an analytic function $f : G \to \mathbb{C}$ is the value n in Proposition 4.5. More general, if $\Omega \subset \mathbb{C}$ is open, $f : \Omega \to \mathbb{C}$ is analytic and $f(y) = 0$, then the proposition applies to the restriction of f to an open ball centered to y and the value n is the order of the zero of f at y.

Proposition 4.6. *Let $D \subset \mathbb{C}$ be a domain. If $D \subset G$ has a limit point in G, then D is a determining set.*

Proof. Suppose f is analytic on G and zero on D and let $a \in D$ be a limit point of D. In particular, $f(y) = 0$. Suppose f is not identically zero. In this case, by Proposition 4.5, there is an n and an analytic function g on G such that $f(z) = (z-y)^m g(z)$ and

$g(y) \neq 0$. On the other hand, g must vanish on $D\backslash\{y\}$ since f does and $(z-y)^m$ does not. Since y is a limit point of D and g is continuous, $g(y) = 0$, a contradiction. □

Corollary 4.7. *The zeros of an non-constant analytic function $f : G \to \mathbb{C}$ are isolated; i.e., if $y \in G$ and $f(y) = 0$, then there an $r > 0$ such that if $z \in G$ and $0 < |z-y| < r$, then $f(z) \neq 0$.*

Proof. Suppose f has a non-isolated zero. In this case there exists a sequence (a_n) from $G\backslash\{y\}$ that converges to y. Hence D, the range of this sequence is a subset of G with a limit point in D. Since f is zero on D, by Propostion 4.6, f is identically 0. □

5. The index

Lemma 5.1. *Suppose $\Omega \subset \mathbb{C}$ is open, $\Pi \subset \mathbb{C}$ and $\psi : \Omega \times \Pi \to \mathbb{C}$. If*

(i) ψ is continuous;
(ii) for each fixed w the function $\psi_w : \Omega \to \mathbb{C}$ defined by $\psi_w(z) = \psi(z,w)$ is analytic;
(iii) there is a constant $C \in \mathbb{R}_{\geqslant 0}$ such that, for all $\zeta \neq z \in \Omega$ and $w \in \Pi$,

$$\left|\frac{\psi(\zeta,w) - \psi(z,w)}{\zeta - z}\right| \leqslant C;$$

(iv) $\gamma : [a,b] \to \Pi$ is a piecewise smooth curve,

then the function $g : \mathbb{C}\backslash\{\gamma\} \to \mathbb{C}$ defined by

$$g(z) = \int_\gamma \psi(z,w)\,dw$$

is analytic and

$$g'(z) = \int_\gamma \frac{\partial\psi}{\partial z}(z,w)\,dw = \int_a^b \frac{\partial\psi}{\partial z}(z,\gamma(s))\gamma'(s)\,ds, \tag{3}$$

In particular, for each z, the function $\rho_z(\gamma(s))\gamma'(s)$ is integrable.

Proof. Without loss of generality, assume γ is smooth. Thus,

$$g(z) = \int_a^b \psi(z,\gamma(s))\gamma'(s)\,ds.$$

Fix z and suppose ζ_n is a sequence from Ω converging to z with $\zeta_n \neq z$. Define $f_n : [a,b] \to \mathbb{C}$ by

$$f_n(s) = \frac{\psi(\zeta_n,\gamma(s)) - \psi(z,\gamma(s))}{\zeta_n - z}\gamma'(s).$$

By the hypothesis of item (iii) and the continuity of γ', the sequence (f_n) is uniformly bounded. By item (ii) it converges pointwise on $[a,b]$ to the function

$$f(s) = \psi'_{\gamma(s)}(z)\gamma'(s) = \frac{\partial\psi}{\partial z}(z,\gamma(s))\gamma'(s).$$

In particular, f is integrable. By Theorem 1.1 (bounded convergence theorem),

$$\lim_n \frac{g(\zeta_n) - g(z)}{\zeta_n - z} = \lim_n \int_a^b f_n(s)\, ds = \int_a^b f(s)\, ds$$

and hence g is differentiable and its derivative is given by equation (3). By Theorem 3.2 (Goursat's Theorem), g is analytic. □

Suppose $\gamma : [a, b] \to \mathbb{C}$ is a (continuous) curve and let $U = \mathbb{C}\backslash\{\gamma\}$. Since U is open, its connected components are open sets. Further, there is an $R > 0$ such that $\{\gamma\} \subset B(0; R)$. Hence, $\{|z| > R\}$ lies in a single component called the *unbounded component* of $\mathbb{C}\backslash\{\gamma\}$.

Proposition 5.2. *If $\gamma : [a, b] \to \mathbb{C}$ is a closed rectifiable path and $\psi : \{\gamma\} \to \mathbb{C}$ is continuous, then $f : \mathbb{C}\backslash\{\gamma\} \to \mathbb{C}$ defined by*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{\psi(w)}{w - z}\, dw$$

is analytic.

If $\gamma : [a, b] \to \mathbb{C}$ is a closed rectifiable path, then the function $n_\gamma : \mathbb{C}\backslash\{\gamma\} \to C$ defined by

$$n_\gamma(y) = \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - y}\, dw$$

takes integer values, is constant on components and is zero on the unbounded component.

Proof. Suppose, for now γ is piecewise smooth. By Lemma 5.1 applied to $\psi : (\mathbb{C}\backslash\{\gamma\}) \times \{\gamma\} \to \mathbb{C}$ defined by $\psi(w) = \frac{1}{2\pi i}\frac{1}{w-z}$, the function n_γ is analytic on $\mathbb{C}\backslash\{\gamma\}$ and, in particular, continuous. To show that n_γ takes integer values. Fix $y \in \mathbb{C}\backslash\{\gamma\}$ and define $g : [a, b] \to \mathbb{C}$ by

$$g(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - y}\, ds.$$

By the second fundamental theorem of calculus, g is differentiable and

$$g'(t) = \frac{\gamma'(t)}{\gamma(t) - y}.$$

Thus,

$$(\gamma - y)' - g'(\gamma - y) = 0.$$

This first order linear differential equation has solution,

$$\gamma - y = \exp(-\int g') = Ce^{-g},$$

for some $C \in \mathbb{C}$. Observe $g(a)$ and $g(b)$ must differ by an integer multiple of $2\pi i$ since $\gamma(a) = \gamma(b)$. Since $g(0) = 0$, we conclude that $g(b)$ is an integer multiple of $2\pi i$.

For the general case, given $\epsilon > 0$, choose a piecewise smooth path Γ such that $y \in \mathbb{C}\backslash\{\Gamma\}$ and

$$|n_\gamma(y) - n_\Gamma(y)| < \epsilon.$$

It follows that $n_\gamma(y)$ is an integer multiple of $2\pi i$. $\square$

The number $n_\gamma(y)$ is the *index* or *winding number* of γ with respect to a. Recall $-\gamma$ is the function $-\gamma : [-b,-a] \to \mathbb{C}$ defined by $-\gamma(s) = \gamma(-s)$. From properties of line integrals,

$$n_{-\gamma}(y) = -n_\gamma(y).$$

Likewise, if γ and δ are equivalent paths, then

$$n_\gamma(y) = n_\delta(y).$$

Finally, given two paths $\gamma : [a,b] \to \mathbb{C}$ and $\delta : [b,c] \to \mathbb{C}$, let $\gamma + \delta$ denote the path with domain $[a,c]$ defined in the obvious way and note,

$$n_{\gamma+\delta}(y) = n_\gamma(y) + n_\delta(y).$$

Alternately, define $n_{\gamma+\delta}(y)$ by this formula.

6. Cauchy's Theorem and integral formula

Given an open set $\Omega \subset \mathbb{C}$, an analytic function $f : \Omega \to \mathbb{C}$ and closed rectifiable curves γ_j in Ω, let $\gamma = \sum \gamma_j$ and define, for $w \in \mathbb{C}\backslash\{\gamma\}$,

$$n_\gamma(w) := \sum_{j=1}^{N} n_{\gamma_j}(w)$$

and

$$\int_\gamma \frac{f(w)}{w-y}\, dw := \sum_{j=1}^{N} \int_\gamma \frac{f(w)}{w-y}\, dw.$$

The curve γ is *homologous to zero (in Ω)* if $n_\gamma(w) = 0$ for all $w \in \mathbb{C}\backslash\Omega$.

Theorem 6.1. *Suppose $\Omega \subset \mathbb{C}$ is open, $f : \Omega \to \mathbb{C}$ is analytic. If $N \in \mathbb{N}^+$ and $\gamma_j : [a,b] \to \Omega$ are closed rectifiable paths such that $\gamma = \sum \gamma_j$ is homologous to zero, then for $y \in \mathbb{C}\backslash\{\gamma\}$,*

$$n_\gamma(y) f(y) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-y}\, dw.$$

Lemma 6.2. *If $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$ is analytic, then the function $\psi : \Omega \times \Omega \to \mathbb{C}$ defined by*

$$\psi(z,w) = \begin{cases} \frac{f(z)-f(w)}{z-w} & z \neq w \\ f'(w) & z = w \end{cases} \tag{4}$$

is continuous and has continuous partial derivatives. †

Proof. Let $\Delta = \{(z,z) : z \in \Omega\}$. Continuity of ψ and of $\frac{\partial \psi}{\partial z}$ on $(\Omega \times \Omega)\backslash\Delta$ is not in doubt. Fix $(y,y) \in \Delta$ and choose $R > 0$ such that $\overline{B(y;2R)} \subset \Omega$. Let M denote a bound for

$|f|$ on $\overline{B(y;2R)}$. For $w \in B(y;R)$, Cauchy's estimate (Corollary 2.2 (iv)) implies (since $B(w;R) \subset B(y;2R)$),

$$|f^{(k)}(w)| \leqslant \frac{k!M}{(R)^k}.$$

Fix $0 < r < \frac{R}{2}$ and let $X = B(y;r) \times B(y;r)$. For $(z,w) \in X$, the sequence of partial sums,

$$S_N(z,w) = \sum_{k=0}^{N} \frac{f^{(k)}(w)}{k!}(z-w)^k$$

converges uniformly to $f(z)$ on X. Moreover, the sequence of partial sums

$$\sum_{k=1}^{N} \frac{f^{(k)}(w)}{k!}(z-w)^{k-1}$$

converges uniformly on X and hence to a continuous function $F : X \to \mathbb{C}$ with continuous partial derivatives. For $z \neq w$,

$$F(z,w) = \lim_N \frac{S_N(z,w) - f(w)}{z-w} = \psi(z,w)$$

and for $z = w$, $F(z,z) = f'(z) = \psi(z,z)$. □

Proof Theorem 6.1 for piecewise smooth γ. Suppose γ is piecewise smooth. Since $n_\gamma : \mathbb{C}\backslash\{\gamma\} \to \mathbb{C}$ is continuous and integer valued, the set $H = \{w \in \mathbb{C}\backslash\{\gamma\} : n_\gamma(w) = 0\}$ is open. By hypothesis $\mathbb{C} = \Omega \cup H$.

By Lemma 5.1 the function $g_1 : H \to \mathbb{C}$ defined by

$$g_1(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-\zeta}\,dw$$

is analytic. Define $\psi : \Omega \times \Omega \to \mathbb{C}$ as in equation (4). It is an exercise to show, for $K \subset \Omega$ is compact, that Lemma 6.2 implies the hypotheses of Lemma 5.1 are satisfied by ψ restricted to $K \times \{\gamma\}$. Hence $g_2 : \Omega \to \mathbb{C}$ defined by

$$g_2(\zeta) = \int_\gamma \psi(\zeta,w)\,dw$$

is analytic. If $z \in H \cap \Omega$, then

$$g_2(z) = \int_\gamma [\frac{f(w)-f(z)}{w-z}]\,dw = g_1(z)$$

because

$$\int_\gamma \frac{f(z)}{w-z}\,dw = f(z)\int_\gamma \frac{1}{w-z}\,dw = f(z)n_\gamma(z) = 0.$$

Hence g_1 and g_2 determine an entire function g. On the other hand, $\lim_{z\to\infty} g(z) = 0$ from the definition of g_1. Hence g is bounded and entire and hence constant by Theorem 4.2.

Since $\lim_{z\to\infty} g(z) = 0$ this constant is 0 and the desired formula holds on H. Returning to g_2, for $z \in \Omega\backslash\{\gamma\}$,

$$\begin{aligned} 0 &= \int_\gamma \frac{f(w)-f(z)}{w-z}\,dw \\ &= \int_\gamma \frac{f(w)}{z-w}\,dw - f(z)\int_\gamma \frac{1}{w-z}\,dw \\ &= \int_\gamma \frac{f(w)}{z-w}\,dw \;-\; 2\pi i n_\gamma(z) f(z). \end{aligned}$$

Finally to drop the piecewise smooth hypothesis, fix a $y \in \mathbb{C}\backslash\{\gamma\}$ and choose an open set U such that $K = \overline{U} \subset \Omega\backslash\{y\}$. Note the collection of functions $F_z : K \to \mathbb{C}$ defined by $F_z(w) = \frac{1}{z-w}$ together with the function $f(w)z - w$ is equicontinuous and apply Lemma 39.1 □

Corollary 6.3. *Under the hypotheses of Theorem 6.1, for each $k \in \mathbb{N}$ and $w \in \mathbb{C}\backslash\{\gamma\}$,*

$$n_\gamma(w) f^{(k)}(w) = \frac{k!}{2\pi i}\int_\gamma \frac{f(z)}{(z-w)^{k+1}}\,dz.$$

†

Sketch of proof. Verify that it is permissible to differentiate under the integral sign in Theorem 6.1. □

Theorem 6.4 (Cauchy's Theorem)**.** *Suppose $\Omega \subset \mathbb{C}$ is open and $g : \Omega \to \mathbb{C}$ is analytic, $N \in \mathbb{N}^+$ and γ_j are rectifiable curves with $\{\gamma_j\} \subset \Omega$ for $j = 1, 2, \dots, N$. Let $\gamma = \sum \gamma_j$. If $n_\gamma(w) = 0$ for each $w \in \mathbb{C}\backslash\Omega$ (γ is homologous to zero in Ω), then*

$$\int_\gamma g\,dz = 0.$$

Proof. Fix $y \in \mathbb{C}\backslash\{\gamma\}$ and apply Theorem 6.1 to the function $f = g(z)(z-y)$. □

7. Homotopic curves and Cauchy's Theorem

7.1. **Homotopy.** Let $\Omega \subset \mathbb{C}$ be an open set. Closed rectifiable paths $\gamma, \delta : [0,1] \to \Omega$ are *homotopic* in Ω, if there exists a continuous function $\Gamma : [0,1] \times [0,1] \to \Omega$ such that

(i) $\Gamma(s, 0) = \gamma(s)$;
(ii) $\Gamma(s, 1) = \delta(s)$;
(iii) $\Gamma(0, t) = \Gamma(1, t)$ for all t.

Remark 7.1. The definition makes sense without the rectifiable assumption. On the other hand, even assuming γ and δ are rectifiable, there is not the assumption that the closed curves $\gamma_t(s) = \Gamma(s, t)$ are rectifiable for $0 < t < 1$. Homotopy defines an equivalence relation on closed (rectifiable) paths in Ω. ⋄

An open set $\Omega \subset \mathbb{C}$ is *star shaped* if there exists a point $y \in \Omega$ such that the line segment $[\![y, z]\!]$ lies in Ω for each $z \in \Omega$. In this case, Ω is *star shaped with respect to* y.

Proposition 7.2. *If $\Omega \subset \mathbb{C}$ is star shaped, then Ω is connected and every pair of closed paths in Ω are homotopic.* †

Proof. Suppose Ω is star shaped with respect to y. Given $z, w \in \Omega$, the path $[\![z, y, w]\!] = [\![z, y]\!] + [\![y, w]\!]$ lies in Ω. Hence Ω is path connected and therefore connected.

To prove the second statement, it suffices to show that every curve $\gamma \in \Omega$ is homotopic to the curve $\delta : [0, 1] \to \Omega$ defined by $\delta(s) = y$. To this end, define $\Gamma : [0, 1] \times [0, 1] \to \Omega$ by $\Gamma(s, t) = t\gamma(s) + (1 - t)y$. □

As an example, consider the non-rectifiable closed path $\gamma : [0, 2] \to \mathbb{C}$ defined by $\gamma(t) = t + it\sin(\frac{2\pi}{t})$ for $1 \leqslant t > 0$ and $f(0) = 0$ and $\gamma(t) = t - 1 + 0i$ for $1 \leqslant t \leqslant 2$. Choosing $R \in \mathbb{R}_{>0}$ large enough so that $\{\gamma\} \in B(0; R)$, by Proposition 7.2, γ is homotopic to each constant curve in $B(0; R)$. Thus, we can choose two constant curves and construct a homotopy Γ between them so that $\Gamma(s, \frac{1}{2}) = \gamma$ is not rectifiable.

A curve γ in Ω that is homotopic to a constant curve is *homotopic to* 0.

7.2. Cauchy Thoerem for homotopic curves.

Theorem 7.3 (Homotopic version of Cauchy's Theorem)**.** *Suppose $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$ is analytic. If $\gamma, \delta : [0, 1] \to \Omega$ are homotopic closed rectifiable paths, then*

$$\int_\gamma f\,dz = \int_\delta f\,dz.$$

If $\sigma, \tau : [0, 1] \to \Omega$ are rectifiable curves and there is an open ball $B \subset \Omega$ such that the path

$$\Delta = \sigma + [\![\sigma(1), \tau(1)]\!] - \tau - [\![\sigma(0), \tau(0)]\!]$$

lies in B, then by Corollary 2.2(vi), $\int_\Delta f = 0$. Equivalently,

$$\int_\sigma f - \int_\tau f = \int_{\sigma(0)}^{\tau(0)} f - \int_{sigma(1)}^{\tau(1)} f.$$

The proof of Theorem 7.3 involves repeated applications of this observation.

Lemma 7.4. *Suppose $\Omega \subset \mathbb{C}$ is open, $f : \Omega \to \mathbb{C}$ is analytic, $\gamma : [0, 1] \to \Omega$ is rectifiable and $P = \{0 = s_0 < s_1 \cdots < s_N = 1\}$ is a partition of $[0, 1]$. Let $\gamma_j = \gamma|_{[s_{j-1}, s_j]}$ for $1 \leqslant j \leqslant N$ and let σ_j denote the path $\gamma_j + [\![\gamma(s_j), \gamma(s_{j-1})]\!]$. Let τ denote the polygonal path $[\![\gamma(s_0), \gamma(s_1), \ldots, \gamma(s_n)]\!]$. If for each $1 \leqslant j \leqslant N$ there exists an open ball $B_j = B_j(y_j; r_j)$ such that $B_j \subset \Omega$ and σ_j lies in B_j, then τ lies in Ω and*

$$\int_\gamma f\,dz = \int_\tau f\,dz.$$

†

Proof. By assumption, each line segment $L_j = [\![\gamma(s_{j-1}), \gamma(s_j)]\!]$ lies in Ω. Hence so does the polygonal path τ.

Note that each τ_j is a closed path in $B_j \subset \Omega$. Hence, using Corollary 2.2 (vi) in the last equality,

$$\begin{aligned}\int_\gamma f\,dz - \int_\tau f\,dz &= \sum \int_{\gamma_j} f\,dz - \int_{L_j} f\,dz \\ &= \sum \int_{\sigma_j} f\,dz = 0.\end{aligned}$$

□

Lemma 7.5. *Suppose $\Omega \subset \mathbb{C}$ is open, $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is analytic, $\tau, \sigma : [0,1] \to \mathbb{C}$ are rectifiable and $P = \{0 = s_0 < s_1 \cdots < s_N = 1\}$ is a partition of $[0,1]$. For $1 \leqslant j \leqslant N$, let $I_j = [s_{j-1}, s_j]$ and for $1 \leqslant 0 \leqslant j \leqslant N$ let L_j denote the line segment $[\![\gamma(s_j), \delta(s_j)]\!]$. Let Δ_j denote the path*

$$\Delta_j = \tau|_{I_j} + L_j - \sigma|_{I_j} - L_{j-1}.$$

If for each $1 \leqslant j \leqslant N$, there is a ball $B_j \subset \Omega$ such that Δ_j lies in B_j, then

$$\int_\tau f\,dz = \int_\sigma f\,dz.$$

†

Proof. Note that each Δ_j is a closed rectifiable path lying in $B_j \subset \Omega$ and $\sum \Delta_j = \gamma - \delta$. Hence, using Corollary 2.2 (vi)

$$\int_\tau f\,dz - \int_\sigma f\,dz = \sum \int_{\Delta_j} f\,dz = 0.$$

□

Proof of Theorem 7.3. Since Γ is continuous with compact domain, its range is a compact subset of Ω. Hence there exists an $r > 0$ such that if x is in the range of Γ, then $B(x; r) \subset \Omega$. Since Γ is uniformly continuous, there exists a positive integer n such that if $(s,t), (u,v) \in [0,1] \times [0,1]$ and $\|(s,t) - (u,v)\| < \frac{2}{n}$, then

$$|\Gamma(s,t) - \Gamma(u,v)| < r.$$

Let P denote the partition of $[0,1]$ determined by the points $s_j = \frac{j}{n}$ and, for $1 \leqslant j \leqslant n$. For $0 \leqslant j \leqslant n$, let γ_j denote the polygonal path

$$\gamma_j = [\![\Gamma(s_0, s_j), \Gamma(s_1, s_j), \dots, \Gamma(s_n, s_0)]\!].$$

Let $\Delta_{j,k} = [\![\Gamma(s_{k-1}, s_{j-1}), \Gamma(s_k, s_{j-1}), \Gamma(s_k, s_j), \Gamma_{s_{k-1}, s_j}]\!]$ and note that $\Delta_{j,k}$ is a closed polygonal path whose vertices lie in the ball $B_{j,k} = B(\Gamma(s_j, s_k); r) \subset \Omega$. Hence, by Lemma 7.5,

$$\int_{\gamma_{j-1}} f\,dz = \int_{\gamma_j} f\,dz.$$

Now consider the curves γ and γ_0 and fix a $1 \leqslant k \leqslant n$. For $s_{k-1} \leqslant s \leqslant s_k$,

$$|\Gamma(s_k, s_0) - \gamma(s)| = |\Gamma(s_k, s_0) - \Gamma(s, s_0)| < r,$$

since $|(s_k, s_0) - (s, s_0)| < \frac{2}{n}$. Likewise, for some $0 \leqslant \lambda \leqslant 1$ depending on s,

$$|\Gamma(s_k, s_0) - \gamma_0(s)| = |\lambda(\Gamma(s_k, s_0) - \Gamma(s_k, s_0) + (1-\lambda)(\Gamma(s_k, s_0) - \Gamma(s_0, s_0))| < r.$$

Thus the path $\sigma_j = \gamma|_{[s_{k-1}, s_k]} - [\![\gamma(s_{k-1}), \gamma(s_k)]\!]$ lies in the ball $B(\Gamma(s_k, s_0); r) \subset \Omega$. Hence, by Lemma 7.4,

$$\int_\gamma f\, dz = \int_{\gamma_0} f\, dz$$

and likewise for δ and γ_n. The conclusion of the theorem now follows. □

We now collect several corollaries of Theorem 7.3.

Corollary 7.6. *If $\Omega \subset \mathbb{C}$ is open, γ is a closed rectifiable path homotopic to 0 in Ω and $f : \Omega \to \mathbb{C}$ is analytic, then*

$$\int_\gamma f\, dz = 0.$$

In particular, γ is homologous to 0 in Ω ($n_\gamma(w) = 0$ for all $w \in \mathbb{C}\backslash\Omega$). †

Proof. The first statement follows immediately from Theorem 7.3. The second follows from the first by choosing $f(z) = \frac{1}{w-z}$ and noting f is analytic in Ω (since $w \notin \Omega$). □

The Corollary 7.6 says, in part, if γ is homotopic to 0, then γ is homologous to 0 (in Ω). The converse is not necessarily true.

7.2.1. *Independence of path.* Suppose $\Omega \subset \mathbb{C}$ is open. Rectifiable curves $\gamma, \delta : [0,1] \to \Omega$ are *(fixed endpoint) homotopic (in Ω)* if $\gamma(0) = \delta(0)$, $\gamma(1) = \delta(1)$ and there exists a continuous function $\Gamma : [0,1] \times [0,1] \to \Omega$ such that

(1) $\Gamma(0, t) = \gamma(0)$ for all t;
(2) $\Gamma(1, t) = \gamma(1)$ for all t;
(3) $\Gamma(s, 0) = \gamma(s)$; and
(4) $\Gamma(s, 1) = \delta(s)$.

Remark 7.7. An exercise shows if γ and δ are (rectifiable and) homotopic, then $\gamma - \delta$ is a (rectifiable) closed curve homotopic to 0. ◇

Corollary 7.8. *If $\Omega \subset \mathbb{C}$ is open, $f : \Omega \to \mathbb{C}$ is analytic and γ, δ are homotopic rectifiable curves in Ω, then*

$$\int_\gamma f\, dz = \int_\delta f\, dz.$$

†

7.2.2. *simple connectedness.*

Proposition 7.9. *If $G \subset C$ is open and connected and if $y, z \in G$, then there is a rectifiable curve $\gamma : [0,1] \to G$ such that $\gamma(0) = y$ and $\gamma(1) = z$.* †

Sketch of proof. The proof that connected plus locally path connected is easily modified to prove this proposition when one uses instead that an open set $\Omega \subset \mathbb{C}$ is *locally rectifiable path connected.* □

An open set $\Omega \subset \mathbb{C}$ is *simply connected* if it is connected and every closed rectifiable path in Ω is homotopic to 0. Thus a star shaped domain is simply connected. In particular, an open ball is simply connected. Item (i) of Corollary 7.10 below thus contains Corollary 2.2(vi) (which was used in the proof of Theorem 7.3) as a special case.

Corollary 7.10. *Suppose G is a simply connected and $f : G \to \mathbb{C}$ is analytic.*

(i) If $\gamma : [0,1] \to G$ is a closed rectifiable path, then
$$\int_\gamma f\, dz = 0.$$
(ii) If $\gamma, \delta : [0,1] \to G$ are rectifiable paths such that $\gamma(0) = \delta(0)$ and $\gamma(1) = \delta(1)$, then
$$\int_\gamma f = \int_\delta f.$$
(iii) f has a primitive.
(iv) If f never vanishes, then there exists an analytic function $g : G \to \mathbb{C}$ such that $f = e^g$.
(v) If f never vanishes, then f has a square root in G.

†

Proof. Item (i) follows immediately from Corollary 7.6 and the definitions. Item (ii) follows from item (i) by considering the closed rectifiable curve $\gamma - \delta$. To prove item (iii), fix $y \in G$. Define a function $F : G \to \mathbb{C}$ as follows. Given $z \in G$ choose, using Proposition 7.9, a rectifiable curve $\gamma : [0,1] \to G$ such that $\gamma(0) = y$ and $\gamma(1) = z$ and define
$$F(z) = \int_\gamma f.$$
By item (ii), F is well defined. Now fix a point w and an $r > 0$ such that $B(w;r) \subset G$. Let γ be a path from y to w and note, for $z \in B(w;r)$,
$$F(z) = \int_{\gamma + [\![w,z]\!]} f = \int_\gamma f + \int_w^z f.$$
By Theorem 3.1, $\int_w^z f$ is analytic in $B(y;r)$ and its derivative is f. Hence F is analytic and $F' = f$.

To prove item (iv), note that the function $h = \frac{f'}{f}$ is analytic in G. Hence by item (iii), there is an analytic function $g : G \to \mathbb{C}$ such that $g' = h$. Observe

$$(f\exp(-g))' = f'\exp(-g) - \exp(-g)f' = 0.$$

Hence there is a non-zero constant $c \in \mathbb{C}$ such that $f = c\exp(g)$. In particular, choosing d so that $d^2 = c$, the function $h = d\exp(\frac{1}{2}g)$ is analytic and $h^2 = f$, proving item (v). □

Remark 7.11. Note, in item (iv), f satisfies $f' - g'f = 0$. Formally, this first order linear equation has solution,

$$f = \exp(\int g') = c\exp(g),$$

motivating the choice of g in the proof. ◇

Remark 7.12. We will later see if any of the conclusions of Corollary 7.10 implies G is simply connected. ◇

8. Counting zeros

The *multiplicity* of a zero y of an analytic function f is a synonym for the order of the zero.

Proposition 8.1. *Suppose $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$ is analytic with finitely many zeros $y_1, \dots y_r$ (counted with multiplicity). If $\gamma : [0,1] \to \mathbb{C}$ is a closed rectifiable curve that is homologous to zero and if $\{\gamma\} \cap \{y_1, \dots, y_r\} = \emptyset$, then*

$$\sum_{j=1}^{r} n_\gamma(y_j) = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}dz.$$

†

There is nothing special here about 0 and $f(z) = 0$. Given $y \in \mathbb{C}$, simply replace f by $f - y$ in Proposition 8.1 to count the zeros of the equation $f(z) = y$.

Proof. By Proposition 4.5, there is an analytic function $g : \Omega \to \mathbb{C}$ such that g is never 0 and

$$f(z) = \prod_{j=1}^{s}(z - y_j)g(z).$$

It follows that

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{s}\frac{1}{z - y_j} + \frac{g'(z)}{g(z)}.$$

Since $\frac{g'}{g}$ is analytic in Ω and, by assumption, γ is homologous to zero in Ω, an application of Theorem 6.4 and Proposition 5.2 hows

$$\int_\gamma \frac{g'(z)}{g(z)}\,dz = 0$$

and the proof is complete. □

8.1. **The Open Mapping Theorem.** A zero y of an analytic function f is *simple* if it has order one.

Proposition 8.2. *If $\Omega \subset \mathbb{C}$ is open, $f : \Omega \to \mathbb{C}$ is analytic and $y \in \Omega$ is a zero of f of order m, then there exists $r, \delta > 0$ such that $B(y; r) \subset \Omega$ and if $0 < |w| < \delta$, then $g(z) = f(z) - w$ has exactly m zeros in $B(y; r)$ and they are all simple.* †

Proof. By Theorem 4.6, y can not be an accumulation point for the zeros of f' (as otherwise f' is constant on the component of Ω containing y and hence zero on this component). By Corollary 4.7 y is not an isolated zero of f. Hence there is an $r > 0$ such that $B(y, 2r) \subset \Omega$ and y is the only zero of f in $B(y; 2r)$ and $f'(z) \neq 0$ for $0 < |z - y| < 2r$ too. Let $\gamma : [0, 2\pi] \to \Omega$ denote the curve $\gamma(s) = y + re^{is}$. Let $\sigma = f \circ \gamma$. It follows that $0 \notin \{\sigma\}$ and thus there is a $\delta > 0$ such that $B(0; \delta) \cap \{\sigma\} = \emptyset$. For $w \in B(0; \delta)$, the points 0 and w lie in the same component of $\mathbb{C}\backslash\{\sigma\}$ and therefore, by Proposition 5.2 $n_\sigma(w)$ is constantly equal to $n_\sigma(0)$ on $B(0; \delta)$.

Let $y_1, \ldots, y_d$ denote the zeros of $f(z) - w$ in $B(y; r)$. Since $w \notin \{\gamma\}$, the equation $f(z) - w = 0$ has no solutions on $\{\gamma\}$ and thus there is an $0 < r < r' < 2r$ such that $y_1, \ldots, y_d$ are the zeros of $f(z) - w$ on $B(y; r')$. By Proposition 8.1,

$$\begin{aligned} n_\sigma(w) =& \frac{1}{2\pi i} \int_\sigma \frac{1}{z - w}\, dz \\ =& \frac{1}{2\pi i} \int_0^{2\pi} \frac{1}{f(\gamma(s)) - w} f'(\gamma(s))\gamma'(s)\, ds \\ =& \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - w}\, dz \\ =& \sum_{j=1}^{d} n_\gamma(y_j) = d. \end{aligned}$$

Hence $n_\sigma(w) = d$ for all $w \in B(0; \delta)$. Choosing $w = 0$ shows $d = m$. Finally, since $f'(z) \neq 0$ on $B(y; r)$, all the zeros of $f(z) - w$ in $B(y; r)$ are simple. □

Proposition 8.3. *If $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$ is analytic and one-one, then $f'(z) \neq 0$ for $z \in \Omega$.* †

Proof. Arguing the contrapositive, suppose $y \in \Omega$ and $f'(y) = 0$. Let $g(z) = f(z) - f(y)$. It follows that g has a zero of order at least two at y. In particular, by Proposition 8.1, there exists a $w \neq 0$ such that $g(z) = w$ has at least two solutions. It follows that $f(z) = f(y) + w$ has at least two solutions and thus f is not one-one. □

Theorem 8.4 (Open Mapping)**.** *If $G \subset \mathbb{C}$ is a domain and $f : G \to \mathbb{C}$ is analytic and not constant and $U \subset G$ is open, then $f(U)$ is open. In particular, $f(G)$ is open.*

Assuming $V \subset \mathbb{R}^2$ is open, if $F : V \to \mathbb{R}^2$ is continuously differentiable and its derivative is pointwise invertible, then $F(V)$ is open by the inverse function theorem. Thus the open mapping theorem is perhaps stronger than one might expect as there is no need to assume f' never zero.

Proof. Let $\zeta \in f(U)$ be given. Thus there is a point $y \in U$ such that $f(y) - \zeta = 0$. Let g denote the function $g(z) = f(z) - \zeta$. Since f is not constant, g has a zero of finite multiplicity at y. By Proposition 8.2 applied to $g|_U$, there exists an $r, \delta > 0$ such that $B(y;r) \subset U$ and for each $w \in B(0;\delta)$ the equation $g(z) = w$ has a solution in $B(y;r)$ and hence in U. Thus $B(0;\delta) \subset g(U)$. Equivalently $B(\zeta;\delta) \subset f(U)$ and hence $f(U)$ is open. □

Corollary 8.5. *If G is a domain and $g : G \to \mathbb{C}$ is analytic and one-one, then $f(G) \subset \mathbb{C}$ is open and the inverse of the function $f : G \to f(G)$ defined by $f(z) = g(z)$ is analytic.* †

Proof. Since f maps open sets to open sets, the inverse image of an open set under f^{-1} is open. Thus f^{-1} is continuous. Since also $z = f(f^{-1}(z))$ and f' is never 0 by Proposition 8.3, it follows that f^{-1} is analytic by Proposition 3.2.20 in Conway. □

9. Isolated singularities of analytic functions

9.1. **Removable Singularities.** Suppose $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$ is analytic. A point $y \in \mathbb{C}\backslash\Omega$ is an *isolated singularity* of f, if there is an $R > 0$ such that $\{0 < |z-y| < R\} \subset \Omega$. The isolated singularity y is *removable* if there is an analytic function $g : \Omega \cup \{y\} \to \mathbb{C}$ such that $g|_\Omega = f$. Equivalently, if there exists an $r > 0$ such that $B(y;r)\backslash\{y\} \subset \Omega$ and there is an analytic function $g : B(y;r) \to \mathbb{C}$ such that g and f agree on $B(y;r)\backslash\{y\}$.

Theorem 9.1 (Riemann)**.** *Suppose $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$ is analytic and y is an isolated singularity of f. The following are equivalent.*

(1) y is removable;
(2)
$$\lim_{z\to y}(z-y)\,f(z) = 0;$$
(3) there is an $R > 0$ such that $B(y;R)\backslash\{y\} \subset \Omega$ and $|f|$ is bounded on $B(y;R)\backslash\{y\}$.

Proof. Without loss of generality assume $\Omega = B(y;R)\backslash\{y\}$ for some $R \in \mathbb{R}_{>0}$. Suppose the indicated limit exists and is 0. Define $h : \Omega \cup \{y\} \to \mathbb{C}$ by $h(z) = (z-y)^2 f(z)$ for $z \neq y$ and $h(y) = 0$. It follows that $h|_\Omega$ is analytic. Moreover, an easy computation using $\lim_{z\to y}(z-y)f(z) = 0$ shows h is differentiable at y (and $h'(y) = 0$). Thus h is differentiable on $B(y;R)$ and, by Theorem 3.2, analytic. Since $h(y) = 0 = h'(y)$, by Proposition 4.5, there is an analytic function $g : B(y;R) \to \mathbb{C}$ such that $h(z) = (z-y)^2 g(z)$. It follows that g is analytic and $g = f$ on $B(y;R)\backslash\{y\}$. Hence item (2) implies (1). On the other hand the implications item (1) implies item (3) implies item (2) are evident. □

9.2. **Poles.** Suppose $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$. An isolated singularity y of f is a *pole* if
$$\lim_{z\to y}|f(z)| = \infty.$$
An isolated singularity that is neither removable nor a pole is an *essential singularity.*

Proposition 9.2. *Suppose $G \subset \mathbb{C}$ is a domain, $y \in G$ and f is analytic on $G\backslash\{y\}$. If f has a pole at y, then there is an $m \in \mathbb{N}^+$ and an analytic function $g : G \to \mathbb{C}$ such that,*

$$f(z) = \frac{g(z)}{(z-a)^m}.$$

Moreover, there exists $a_{-1}, \ldots, a_{-m} \in \mathbb{C}$ such that

$$f(z) = \sum_{j=1}^{m} a_{-j}(z-y)^{-j} + \varphi(z)$$

for an analytic function $\varphi : G \to \mathbb{C}$. †

The smallest such m is the *order of the pole* of f at y. In particular, $g(y) \neq 0$ for this choice of m. The expression $\sum_{j=1}^{m} a_{-j}(z-y)^j$ is the *singular part* of f.

Proof. Since f has a pole at y, there exists an $r > 0$ such that if $0 < |z-y| < r$, then $|f(z)| > 1$. Hence f has no zeros on $B(y;r)$. Moreover, because f has a pole at y, the function $h : B(y;r)\backslash\{y\} \to \mathbb{C}$ defined by $h(z) = \frac{1}{f(z)}$ has a removable singularity at a and hence extends, by defining $h(y) = 0$ to an analytic function, still denoted by h, on $B(y;r)$. Since $h(y) = 0$, but h is not identically zero, letting m denote the order of the zero of h at y, there is an analytic function $G : B(y;r) \to \mathbb{C}$ such that $h(z) = (z-a)^m G(z)$ and $G(y) \neq 0$. Since G is never zero, it follows that, $(z-y)^m f(z) = \frac{1}{G(z)}$ on $B(y;r)\backslash\{aa\}$. It follows that $(z-y)^m f(z)$ has a removable singularity at y; i.e., there is an analytic function $g : \Omega \to \mathbb{C}$ such that $g(z) = (z-y)^m f(z)$ for $z \neq y$. Since $g(y) \neq 0$, we see that m is as large as possible.

The moreover statement follows easily from what has already been proved by expanding $g(z)$ in a power series in a neighborhood of y. □

Corollary 9.3. *An isolated singularity y of f is either removable or a pole if and only if there is an $m \in \mathbb{N}$ such that $\lim_{z \to y} f(z) = 0$. Indeed, if m is the smallest such positive integer, then y is removable if and only if $m = 0$ and otherwise is a pole of order m.* †

Sketch of proof. If $\lim_{z\to y}(z-y)^m f(z) = 0$, then $(z-y)^m f(z)$ has a removable singularity at y and hence there is an analytic function g (even at y) such that $g(z) = (z-y)^m f(z)$. Assuming m is as small as possible, $g(y) \neq 0$. In the case $m \geqslant 1$, it follows that $\lim_{z\to y} |f(z)| = \infty$. □

9.3. **Laurent Series and essential singularities.** An *annulus* (centered to the origin) is a subset of $\mathbb{C}$ of the form $\{r < |z| < R\}$ for some $0 < r < R$. Given complex numbers $\{a_n : n \in \mathbb{Z}\}$, the expression $\sum_{n=-\infty}^{\infty} a_n z^n$ is a *Laurent Series.* For a given $z \in \mathbb{C}\backslash\{0\}$ the series *converges absolutely* if both $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} a_{-n} z^{-n}$ converge absolutely. In this case $\sum_{-\infty}^{\infty} a_n z^n$ also denotes $\sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n} z^{-n}$. Given $0 < r < R$, this Laurent series *converges uniformly on compact subsets* of $\{r < |z| < R\}$ if $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} a_{-n} z^{-n}$ converge absolutely on the domains $\{|z| < R\}$ and $\{|z| > r\}$ respectively and, for each $0 < r < r' < R' < R$, these series converge uniformly on $\{|z| \leqslant R'\}$ and on $\{|z| \geqslant r'\}$ respectively.

Proposition 9.4. *Suppose* $0 < r < R$ *and* A *is the annulus* $\{r < |z| < R\}$*. If* $f : A \to \mathbb{C}$ *is analytic, then there exists complex numbers* $\{a_n : n \in \mathbb{Z}\}$ *such that the Laurent series* $\sum_{n=-\infty}^{\infty} a_n z^n$ *converges absolutely and uniformly on compact subsets of* A *to* f*. Moreover the* a_n *are unique and, for any* $r < u < R$*, for each* $n \in \mathbb{Z}$*,*

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}} \, dz,$$

where $\gamma : [0, 2\pi] \to A$ *is given by* $\gamma(s) = ue^{is}$*.* †

Sketch of proof. Here is a sketch of the proof, leaving many details to be filled in and assertions to be justified by the gentle reader. That the a_n are independent of γ is an immediate consequence of the homotopic version of Cauchy's Theorem.

Fix $r < u < U < R$ and define $\gamma, \Gamma : [0, 2\pi] \to A$ by $\gamma(s) = u\exp(is)$ and $\Gamma(s) = U\exp(is)$. Let $\tau = \Gamma - \gamma$. Observe,

(i) if $|z| < u$, then $n_\gamma(z) = n_\Gamma(z) = 1$ and $n_\tau(z) = 0$;
(ii) if $|z| > U$, then $n_\Gamma(z) = n_\Gamma(z) = 0$ and $n_\tau(z) = 0$;
(iii) if $u < |z| < U$, then $n_\Gamma(z) = 1$ and $n_\gamma(z) = 0$ and thus $n_\tau(z) = 1$.

From (i) and (ii) it follows that τ is homologous to zero $\mathbb{C}\backslash\{r < |z| < R\}$. Hence, by (iii) and Cauchy's integral formula, if $u < |z| < U$, then using $|z| < |w|$ on Γ and $|z| > |w|$ on γ to see that the series converge uniformly to justify the interchange of summations and integrals,

$$\begin{aligned} f(z) =& \frac{1}{2\pi i} \int_\tau \frac{f(w)}{w - z} \, dw \\ =& \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z} \, dw - \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} \, dw \\ =& \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{1 - \frac{z}{w}} \frac{dw}{w} + \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{1 - \frac{z}{w}} \frac{dw}{z} \\ =& \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w^{n+1}} \, dw \right] z^n + \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_\Gamma f(w) w^n \, dw \right] z^{-n-1}. \end{aligned}$$

To see that the (a_n) are uniquely determined, suppose $\sum_{n=-\infty}^{\infty} b_n z^n$ is a Laurent expansion of f in $A = \{r < |z| < R\}$ with the expansion converging uniformly on compact subsets of A. Using uniform convergence to justify the interchange of limits,

$$a_k = \int_\gamma \frac{f(z)}{z^{k+1}} \, dz = \sum_n \int_\gamma \frac{b_n}{z^{k-n+1}} \, dz = b_k.$$

□

Corollary 9.5. *Suppose* $\Omega \subset \mathbb{C}$ *is open and* $f : \Omega \to \mathbb{C}$ *is analytic. If* y *is an isolated singularity of* f *and*

$$\sum_{n=0}^{\infty} a_n (z - y)^n$$

is the Laurent expansion of f in some annulus in Ω centered to y, then

(1) y is an isolated singularity if and only if $a_n = 0$ for $n < 0$;
(2) y is a pole of order m if and only if $a_n = 0$ for $n < m$; and
(3) y is an essential singularity if and only if $a_n \neq 0$ for infinitely many $n < 0$.

†

Proof. Item (1) is left as an exercise - and is essentially (pun intended) contained in item (2). To prove item (2), suppose y is a pole of f of order m. By proposition 9.2, there is an analytic $g : \Omega \to \mathbb{C}$ such that $f(z) = (z-y)^m g(z)$. Expanding g in a power series (near y) gives a Laurent expansion (and hence the Laurent expansion) $\sum a_n z^n$ for f where $a_n = 0$ for $n < m$. The converse is evident.

Item (3) follows immediately from items (2) and (1) (since an essential singularity, by definition is one that is neither removable nor a pole). □

An isolated singularity y of f is essential if and only if $\lim_{z\to y} |f(z)|$ does not exist as an element of $[0, \infty]$.

Theorem 9.6 (Casorati-Weierstrass)**.** *Suppose $\Omega \subset \mathbb{C}$ is open, $y \in \Omega$ and $f : \Omega\backslash\{y\} \to \mathbb{C}$ is analytic. If f has an essential singularity at y, then for each $0 < r < 1$ such that $B(y; r) \subset \Omega$, the closure of the set $f(\{0 < |z-y| < r\})$ is $\mathbb{C}$.*

Proof. We prove the contrapositive. Accordingly suppose there is an $r > 0$, a point $\zeta \in \mathbb{C}$ and an $\epsilon > 0$ such that $f(\{0 < |z-y| < r\}) \cap \{|z-\zeta| < \epsilon\} = \varnothing$.; i.e, if $0 < |z-y| < r$, then $|f(z)-\zeta| \geqslant \epsilon$. It follows that the function $g(z) = \frac{f(z)-\zeta}{z-y}$ has a pole at y. Let $m \geqslant 1$ denote the order of this pole. Hence, by Corollary (9.3), $\lim_{z\to y}(z-y)^{m+1}g(z) = 0$. Equivalently, $\lim_{z\to y}(z-y)^m(f(z)-\zeta) = 0$. Thus $\lim_{z\to y}(z-y)^m f(z) = 0$ too and consequently f has either a removable singularity (if $m = 1$) or a pole at y. □

Example 9.7. From Corollary 9.5, to exhibit a function with an essential singularity at 0 it suffices to write down a series $\sum_{n=0}^{\infty} a_n z^n$ with at most finitely many terms a_n and for which there is an $r > 0$ such that the series converges absolutely for $0 < |z|$. As a concrete example, consider $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ defined by $f(z) = \exp(\frac{1}{z})$. It has the series expansion $\sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$. It is also easy to see, for each $r > 0$, the set $f(\{0 < |z| < r\}) = \mathbb{C}\backslash\{0\}$ and thus conclude 0 is an essential singularity by Casorati-Weierstrass. △

Problem 9.1. Fix $r > 0$. Suppose (a_n) is sequence from $\mathbb{C}\backslash\{0\}$ that converges to 0 with $|a_n| < r$ and let $G = \{0 < |z| < r\}\backslash\{a_k : k\}$. Show, if f is analytic on G with poles at the points a_k, then for every $w \in \mathbb{C}$ there is a sequence (λ_k) from G converging to 0 with $f(\lambda_k) = w$.

Problem 9.2. Suppose f is analytic on $\{0 < |z| < r\}$. Show, if f has an essential singularity at 0, then for every $c \in \mathbb{C}$, every $\epsilon > 0$ and every $\delta > 0$, there exits a y such that $|c-y| < \epsilon$ and $f(z) = y$ has infinitely many solutions in $\{0 < |z| < \delta\}$. [Suggestion: Consider $f(\{0 < |z| < \frac{1}{n})$ and apply the Baire Category Theorem.]

Problem 9.3. Suppose f is entire and n is a positive integer. Show, if there is an $R, M > 0$ such that $|f(z)| \leqslant M|z|^n$ for $|z| > R$, then f is a polynomial of degree at most n.

Problem 9.4. Suppose f is entire. Show, if $g(z) = f(\frac{1}{z})$ has either a removable singularity or a pole at 0 (i.e., f has a pole or removable singularity at ∞, then f is a polynomial.

10. Residues

Suppose $\Omega \subset \mathbb{C}$ is open, $y \in \Omega$ and $f : \Omega\backslash\{y\}$ is analytic. Thus there is an $R > 0$ such that f has a Laurent expansion in $\{0 < |z - y| < R\}$,

$$\sum_{n=-\infty}^{\infty} a_n z^n.$$

The residue of f at y is a_{-1} and is denoted $\operatorname{res}(f; y)$. Note that, for $0 < \delta < R$, a nonzero integer m and with $\sigma : [0, 1] \to B(y; R)\backslash\{y\}$ denoting the path $\sigma(s) = y + \delta \exp(2\pi m i s)$,

$$\frac{1}{2\pi i}\int_\sigma f = m \operatorname{res}(f; y). \tag{5}$$

Theorem 10.1 (Residue Theorem)**.** *Suppose $\Omega \subset \mathbb{C}$ is open and $\{y_1, \dots, y_d\} \subset \Omega$. Let $G = \Omega\backslash\{y_1, \dots, y_d\}$. If $f : G \to \mathbb{C}$ is analytic and if $\gamma : [0, 1] \to G$ is a closed rectifiable path that is homologous to 0 in Ω, then*

$$\frac{1}{2\pi i}\int_\gamma f = \sum_{j=1}^{d} n_\gamma(y_j) \operatorname{res}(f; y_j).$$

Proof. Let $m_j = n_\gamma(y_j)$. Choose $r_1, \dots; r_d > 0$ such that the balls $B_j = B(y_j; 2r_j)$ are pairwise disjoint and are subsets of Ω. Let $\gamma_j : [0, 1] \to G$ denote the path $\gamma_k(s) = y_k + r_k \exp(-2\pi m_k i s)$. In particular,

$$n_\gamma(y_j) + \sum_{k=1}^{d} n_{\gamma_k}(y_j) = 0.$$

Further, for $y \notin \Omega$,

$$n_\gamma(y) + \sum_{k=1}^{d} n_{\gamma_k}(y) = 0 + 0 = 0,$$

since γ is, by hypothesis, homologous to 0 in Ω. Thus $\tau = \gamma - \sum_k \gamma_k$ is homologous to zero in G. Since f is analytic in G, Theorem 6.4 (Cauchy's Theorem) implies

$$0 = \int_\tau f = \int_\gamma f + \sum_k \int_{\gamma_k} f = \int_\gamma f - \sum m_k \operatorname{res}(f; y_k).$$

□

Remark 10.2. Often the Residue Theorem is used to calculate integrals in terms of the residues (rather than finding residues by calculating integrals). Suppose f has a pole of order m at y. In this case there is an analytic function g such that $g^{(m)}(y) \neq 0$ and $g(z) = (z-a)^m f(z)$ and

$$\operatorname{res}(f;y) = \frac{1}{(m-1)!} g^{(m-1)}(y).$$

◇

11. The Argument Principle

Suppose $\Omega \subset \mathbb{C}$ is open. A function f is *meromorphic* on Ω if there exits a subset G of Ω such that $f : G \to \mathbb{C}$ is analytic and each point of $\Omega \backslash G$ is either a removable singularity or pole of f. In this case, if K is a compact subset of Ω, then f has finitely many poles in K (exercise) and hence the accumulation points of the set of poles of f do not accumulate in Ω.

Theorem 11.1 (Argument Principle)**.** *Suppose $\Omega \subset \mathbb{C}$ is open, $g : \Omega \to \mathbb{C}$ is analytic and f is meromorphic on Ω with finitely many poles $p_1, \dots, p_N$ and finitely many zeros $q_1, \dots, q_n$ each counted according to multiplicity. If $\gamma : [0,1] \to \Omega \backslash \{p_1, \dots, p_m, q_1, \dots, q_n\}$ is a closed rectifiable curve that is homologous to zero in Ω, then*

$$\frac{1}{2\pi i} \int_\gamma \frac{gf'}{f}\, dz = \sum n_\gamma(q_j) g(q_j) - \sum n_\gamma(p_j) g(p_j)$$

Proof. Let $G = \Omega \backslash \{p_1, \dots, p_m\}$. Hence $f : G \to \mathbb{C}$ is analytic. If f has a pole of order m at y, then there is an analytic function $h : G \cup \{y\} \to \mathbb{C}$ such that $h^{(m)}(y) \neq 0$ and $h(z) = (z-y)^m f(z)$. Thus

$$\frac{f'(z)}{f(z)} = \frac{-m}{z-y} + \frac{h'(z)}{h(z)}.$$

Applying to the poles of f repeatedly, produces an analytic $\psi : \Omega \to \mathbb{C}$ such that

$$\frac{g(z)f'(z)}{f(z)} = \sum \frac{g(z)}{z-q_k} - \sum \frac{g(z)}{z-p_j} + \frac{\varphi'(z)}{\varphi(z)}$$

and the zeros of ψ are precisely those of f, namely $q_1, \dots, q_n$. Suppose ψ has zero of order m at y. There exists a meromorphic function $h : G \to \mathbb{C}$ such that $g(y) \neq 0$ and $\psi(z) = (z-y)^m h(z)$. Thus

$$\frac{psi'(z)}{\psi(z)} = \frac{m}{z-y} + \frac{h'(z)}{h(z)}.$$

Applying this result repeatedly produces an analytic function $\varphi : \Omega \to \mathbb{C}$ without zeros such that

$$\frac{g(z)f'(z)}{f(z)} = \sum \frac{g(z)}{z-q_k} - \sum \frac{g(z)}{z-p_j} + \frac{\varphi'(z)}{\varphi(z)}.$$

Since γ is homologous to zero in Ω, Cauchy's integral formula and Cauchy's Theorem imply

$$\frac{1}{2\pi i}\int_\gamma \frac{gf'}{f}\,dz = \sum_k g(q_k)n_\gamma(q_k) - \sum_j g(p_j)n_\gamma(p_j) + 0.$$

□

Corollary 11.2. *Suppose $\Omega \subset \mathbb{C}$ is open and $\varphi : \Omega \to \mathbb{C}$ is analytic and $B(y;R) \subset \Omega$. If φ is one-one on $B(y;R)$, then $\psi = \varphi|_{B(y;R)} \to \varphi(B(y;R))$ has an analytic inverse ψ^{-1} and moreover, for each $r < R$ and $w \in \varphi(B(y;r)$,*

$$\psi^{-1}(w) = \frac{1}{2\pi i}\int_\gamma \frac{z\varphi'(z)}{\varphi(z)-w}\,dz,$$

where $\gamma : [0,1] \to \Omega$ is the curve $\gamma(s) = y + re^{2\pi i s}$. †

Proof. Without loss of generality, suppose $\Omega = B(y;R)$. By Proposition 8.3, $varphi'(z) \neq 0$ for $z \in B(y;R)$. Fix $0 < r < R$ and $w \in \varphi(B(y;r))$. There is a unique $\zeta \in B(y;R)$ such that $\varphi(\zeta) = w$. In Theorem 11.1, choose $g(z) = z$ and $f(z) = \varphi(z) - w$. In particular f has no poles and only the one zero ζ of multiplicity one in $B(y;r)$. Note too $\{\gamma\} \subset B(y;R)\backslash\{\zeta\}$ and is homologous to zero in $B(y;R)$ and $n_\gamma(\zeta) = 1$. Hence, by Theorem 11.1,

$$\psi^{-1}(w) = \zeta = g(\zeta)n_\gamma(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{gf'}{f} = \frac{1}{2\pi i}\int_\gamma \frac{z\varphi'(z)}{\varphi(z)-w}\,dz.$$

□

Theorem 11.3 (Rouche's Theorem)**.** *Suppose f, g are meromorphic on an open set $\Omega \subset \mathbb{C}$ containing $B(y;R)$ with no zeros or poles in $\{\gamma\}$, where $\gamma : [0,1] \to \Omega$ is given by $\gamma(s) = R\exp(2\pi i\, s)$.. Let $Z(f)$ and $Z(g)$ denote the number of zeros of f and g in $B(y;R)$ counted with multiplicity (order) respectively and let $P(f)$ and $P(g)$ denote the number of poles. If, for each $z \in \{\gamma\}$,*

$$|f(z) + g(z)| < |f(z)| + |g(z)|, \tag{6}$$

then $Z(f) - P(f) = Z(g) - P(g)$.

Proof. The hypotheses imply there is an open set U containing $\{\gamma\}$ on neither f nor g never vanishes. Define $h : U \to \mathbb{C}$ by $h = \frac{f}{g}$. Since f doesn't vanish, h is never 0. The inequality (6) implies the ratio $\frac{f(z)}{g(z)}$ is not a positive real number. Hence h takes values in $\mathbb{C}\backslash[0,\infty)$. Let φ denote a branch of the logarithm on $\mathbb{C}\backslash[0,\infty)$ and note the $\Phi(z) = \varphi(h(z))$ is a primitive for $\frac{h'}{h}$ on U. Hence, by the fundamental theorem of line integrals and Theorem 11.1 (the argument principle),

$$\begin{aligned}0 =&\frac{1}{2\pi i}\int_\gamma \frac{h'}{h}\\ =&\frac{1}{2\pi i}\int_\gamma \left[\frac{f'}{f}-\frac{g'}{g}\right]\\ =&\,(Z(f)-P(f))-(Z(g)-P(g))\,.\end{aligned}$$

□

Remark 11.4. Often the hypothesis of Rouche's Theorem has $|f+g|<|g|$ in place of $|f+g|<|f|+|g|$. ◇

Problem 11.1. Use Theorem 11.3 to give yet another proof of the fundamental theorem of algebra. [Hint: If p is a *monic* polynomial of degree n and if $R>0$ is sufficiently large, then $|\frac{p(z)}{z^n}-1|<1$.]

12. The algebras $A(G)$ and $H^\infty(G)$

Given an open set $\Omega\subset\mathbb{C}$, let $H^\infty(\Omega)$ and denote the bounded analytic functions $f:\Omega\to\mathbb{C}$ and let $A(\Omega)$ denote space of functions $f:\overline{\Omega}\to\mathbb{C}$ such that $f|_\Omega:\Omega\to\mathbb{C}$ is analytic. Both these spaces are algebras and, with the norms

$$\|f\|=\|f\|_\infty=\sup\{|f(z)|: z\in\Omega\},$$

are normed algebras such that

$$\|fg\|\leqslant\|f\|\,\|g\|.$$

It turns out that, appropriately interpreted, these norms are achieved on the boundary of Ω. Let $\partial_\infty\Omega$ denote the boundary of Ω in the case that Ω is bounded; and let it denote boundary of Ω plus $\{\infty\}$ (in the extended complex plane) if Ω is not bounded. For $f:\Omega\to\mathbb{C}$ and y a limit point of Ω, let

$$\limsup_{z\to y} f=\lim_{r\to 0, r>0}\sup\{|f(z)|: z\in\Omega\cap B(y;r)\}.$$

The limit exists since the right hand side is decreasing with r. In particular, if $\rho>\limsup_{z\to y} f$, then there is an $r>0$ such that

$$\sup\{|f(z)|: z\in\Omega\cap B(y;r)\}<\rho.$$

Similarly, in the case Ω is unbounded, let

$$\limsup_{z\to\infty} f=\lim_{R\to\infty}\sup\{|f(z)|: z\in\Omega, |z|>R\}.$$

Theorem 12.1. *Suppose $\Omega\subset\mathbb{C}$ is open (and nonempty) and $f:\Omega\to\mathbb{C}$. Let*

$$L_f=\{\limsup_{z\to y}|f(z)|: y\in\partial_\infty\Omega\}.$$

$f\in H^\infty$ if and only if L_f is bounded and in this case

$$\|f\|_\infty=\sup(L_f).$$

Further, if Ω is a domain (connected) and f is not constant, then $|f(z)| < \|f\|_\infty$ for $z \in \Omega$.

The case where Ω is bounded and $f \in A(\Omega)$ more can be said.

Proposition 12.2. *If $\Omega \subset \mathbb{C}$ is (nonempty) open and bounded and $f \in A(\Omega)$, then*

$$\|f\|_\infty = \max\{|f(z)| : z \in \partial\Omega\}.$$

Further, if Ω is a domain (connected) and f is not constant, then $|f(z)| < \|f\|_\infty$ for $z \in \Omega$. †

Lemma 12.3. *If $\Omega \subset \mathbb{C}$ is open and $C \subset \Omega \subset \mathbb{C}$ is a connected component of Ω, then $\partial C \subset \partial\Omega$.* †

Proof. Suppose $x \in \partial C$. Arguing by contradiction, suppose $x \in \Omega$. Choose a connected neighborhood V of x such that $V \subset \Omega$. Since $x \in \partial C$, it follows that $C \cap V \neq \varnothing$. Hence $C \cup V \subset \Omega$ is connected and therefore $V \cup C \subset C$; i.e., $V \subset C$. But then x is not a boundary point of C, a contradiction. Hence $x \notin \Omega$ and thus $x \in \partial\Omega$. □

Proof. Since Ω is bounded, $\overline{\Omega}$ and $\partial\Omega$ are both compact. Hence,

$$\|f\|_\infty = \max\{|f(z)| : z \in \overline{\Omega}\}.$$

In particular, there is a point $p \in \overline{\Omega}$ such that $|f(p)| \geqslant |f(z)|$ for $z \in \overline{\Omega}$. If $p \in \Omega$, then $|f|_\Omega|$ has a maximum in Ω and hence, by Theorem 2.5 (the maximum modulus theorem), $|f|$ is constant on the connected component Ω_p of Ω containing p. Hence $|f(\zeta)| = |f(p)|$ for $\zeta \in \partial\Omega_p$. Since $\partial\Omega_p \subset \partial\Omega$, the result follows. □

Proof of Theorem 12.1. That $f \in H^\infty(\Omega)$ implies L_f is bounded is immediate. Suppose $f : \Omega \to \mathbb{C}$ and L_f is bounded. Let $M = \sup(L_f)$. Given $\rho > M$, let

$$H_\rho = \{z \in \Omega : |f(z)| > \rho\}.$$

Since $|f|$ is continuous, H_ρ is open. Given $y \in \partial\Omega$, there exists an $r > 0$ such that $|f(z)| < \rho$ for $z \in B(y;r) \cap \Omega$. Thus there is an open set $U \supset \partial\Omega$ such that $H_\rho \cap U = \varnothing$. Hence $\overline{H_\rho} \subset \Omega$. Similarly, if Ω is unbounded, then there is an $R > 0$ such that $|f(z)| < \rho$ for $z \in \Omega$ and $|z| > R$. Thus, in any case, H_ρ is bounded. It follows that $\overline{H_\rho}$ is compact and therefore, by Proposition 12.2 applied to f restricted to H_ρ, assuming H_ρ is nonempty,

$$\rho < \sup\{|f(z)| : z \in H_\rho\} = \max\{|f(z)| : z \in \partial H_\rho\} \leqslant \rho,$$

a contradiction. Hence H_ρ is empty and $|f(z)| \leqslant \rho$ for all $z \in \Omega$ and $\rho > M$. Hence $f \in H^\infty(\Omega)$ and $\|f\|_\infty = M$. □

Remark 12.4. Note that we may view, $A(\Omega) \subset C(\partial\Omega)$ in the case of bonded Ω. When $\Omega = \mathbb{D} = \{|z| < 1\}$ (*the unit disk*), $A(\mathbb{D})$ is known as the disc algebra and $H^\infty(\mathbb{D})$ is known as H^∞. Later we will see that these spaces are in fact complete and hence Banach spaces. ◇

12.1. **Mappings of the disk, Schwarz Lemma.** A complex number c is *unimodular* if $|c| = 1$.

Theorem 12.5 (Schwarz Lemma). *If $f : \mathbb{D} \to \overline{\mathbb{D}}$ is analytic and $f(0) = 0$, then*

(i) $|f(z)| \leqslant |z|$ for all $z \in \mathbb{D}$;
(ii) $|f'(0)| \leqslant 1$;
(iii) and if equality holds in either items (i) or (ii), then there is a unimodular c such that $f(z) = c\,z$.

Proof. By Proposition 4.5, there is an analytic function $g : \mathbb{D} \to \mathbb{C}$ such that $f(z) = zg(z)$. Fix $0 < r < 1$. The function g is analytic on $B(0;r)$, continuous on $\overline{B(0;r)}$ and $|g(\zeta)| \leqslant \frac{1}{r}$ on $\partial B(0;r)$. Hence by Proposition 12.2, if $|z| < r$, then $|g(z)| \leqslant \frac{1}{r}$. Fixing z and letting $r > 1$ tend to 1 gives $|g(z)| \leqslant 1$. Thus, $|f(z)| \leqslant |z|$ for $|z| < 1$. To prove item (ii), note that $|\frac{f(z)}{z}| = |g(z)| \leqslant 1$, for $z \neq 0$.

Now suppose there is a $w \in \mathbb{D}$ such that $|f(w)| = |w|$. It follows that $|g(w)| = 1$ and hence, by maximum modulus, $g(z) = |g(w)|$ for all z. Finally, suppose $|f'(0)| = 1$. In this case $|g(0)| = 1$ and again maximum modulus applied to g shows $f(z) = cz$ for some *unimodular* c. □

Remark 12.6. von Neumann's inequality. ◇

Given $w \in \mathbb{D}$, the mapping

$$\varphi_w(z) := \frac{z - w}{1 - \overline{w}z}$$

is easily seen to give a mapping $\varphi_w : \mathbb{D} \to \mathbb{D}$ that is an automorphism (one-one and onto) by verifying $|\varphi_w(z)| \leqslant 1$ for $z \in \mathbb{D}$ and that φ_{-w} is its inverse. In particular, $|\varphi_w(z)| = 1$ for $|z| = 1$.

Proposition 12.7. *If $f : \mathbb{D} \to \mathbb{D}$ is an analytic automorphism (one-one and onto), then there is a $w \in \mathbb{D}$ and a unimodular c such that*

$$f(z) = c\varphi_w(z) := c\frac{z - w}{1 - \overline{w}z}.$$

†

Proof. Since f is onto, there is a w such that $f(w) = 0$. Hence $g = f \circ \varphi_{-w} : \mathbb{D} \to \mathbb{D}$ and $g(0) = 0$. By the Schwarz Lemma, $|g(z)| \leqslant |z|$ for $z \in \mathbb{D}$. Since $g^{-1} : \mathbb{D} \to \mathbb{D}$, the Schwarz lemma also gives $|z| = |g^{-1}(g(z))| \leqslant |g(z)|$ for $z \in \mathbb{D}$. Hence, again by the Schwarz Lemma, there is a unimodular c such that $g(z) = cz$. The result now follows. □

Problem 12.1. Show, if $f : \mathbb{D} \to \mathbb{D}$ is an analytic automorphism, then f has at most one fixed point or $f(z) = z$.

Problem 12.2. This problem generalizes Problem 12.1. The function $\rho : \mathbb{D} \to \mathbb{D}$ defined by $\rho(z,w) = \frac{|z-w|}{|1-\bar{w}z|}$ is the *Poincare* metric or *hyperbolic distance* between z and w. Show, if $f : \mathbb{D} \to \mathbb{D}$ and $z, w \in \mathbb{D}$, then

$$\rho(f(z), f(w)) \leqslant \rho(z,w).$$

13. The topology of uniform convergence on compact sets

13.1. **Preliminaries.** An *exhaustion* of an open set $\Omega \subset \mathbb{C}$ by compact sets is a sequence (K_n) of compacts sets such that

(i) $\Omega = \cup K_n$;
(ii) for each n, $K_n \subset K_{n+1}^\circ$ (interior).

Lemma 13.1. *If (K_n) is an exhaustion of Ω and $K \subset \Omega$ is compact, then there exists an N such that $K \subset K_N^\circ$.* †

Proof. Observe $\Omega = \cup K_n^\circ$. Hence, if $K \subset \Omega$ is compact, then $\{K_n^\circ : n\}$ is an open cover of K. Since $K_n^\circ \subset K_{n+1}^\circ$, it follows that there is an N such that $K \subset K_N^\circ$. □

Proposition 13.2. *There is an exhaustion (K_n) of Ω such that if $n \in \mathbb{N}$ and C is a connected component of $\mathbb{C}_\infty \backslash K_n$ then C contains a component of $\mathbb{C}_\infty \backslash \Omega$.* †

Proof. Let

$$K_n = \{|z| \leqslant n\} \cap \{z : d(z, \mathbb{C}\backslash\Omega) \geqslant \frac{1}{n}\}.$$

Here $d(z, \mathbb{C}\backslash\Omega)$ is the distance from z to $\mathbb{C}\backslash\Omega$. The set K_n are closed and bounded and hence compact. Let

$$U_n = \{|z| < n\} \cap \{z : d(z, \mathbb{C}\backslash\Omega) > \frac{1}{n}\}$$

and note each U_n is an open subset of Ω. Since $K_n \subset U_{n+1} \subset K_{n+1}$ it follows that $K_n \subset K_{n+1}^\circ$. Since $\Omega = \cup U_n$ we conclude $\Omega \subset \cup K_n^\circ$ too.

Now fix n and suppose C is a connected component of $\mathbb{C}_\infty \backslash K_n$. Since $C_\infty \backslash K_n \supset \mathbb{C}_\infty \backslash \Omega$, the result is true if C is the unbounded component (the component containing ∞) of $\mathbb{C}_\infty \backslash K_n$. Note too, that the unbounded component contains $\{|z| > n\}$. Now suppose C is a bounded component and $z \in C$. In particular, $|z| < n$ and thus $d(z, \mathbb{C}\backslash\Omega) < \frac{1}{n}$. Hence there is a point $w \in \mathbb{C}\backslash\Omega$ such that $|z - w| < \frac{1}{n}$. Now $z \in B(w; \frac{1}{n}) \subset \mathbb{C}\backslash K_n$ as $w \in \mathbb{C}\backslash\Omega$. The set $B(w; \frac{1}{n})$ is connected subset of $\mathbb{C}\backslash K_n$ that contains $z \in C$. Therefore, $B(w; \frac{1}{n}) \subset C$ and in particular $w \in C$. Let C' denote the component of $\mathbb{C}\backslash\Omega$ containing w. Since $w \in C' \subset \mathbb{C}\backslash K_n$ and $w \in C$, it follows that $C' \subset C$. □

Let X be a set. A function $\tau : X \times X \to [0, \infty)$ is a *semi-metric* if it satisfies all the axioms of a metric except for $\tau(x,y) = 0$ does not necessarily imply $x = y$.

Lemma 13.3. *If d is a semimetric on X, then so is $\tau : X \times X \to [0, \infty)$ defined by*

$$\tau(x,y) = \frac{d(x,y)}{1 + d(x,y)}.$$

†

Sketch of proof. Compute, for $a, b \geqslant 0$,

$$\frac{a+b}{1+a+b} - \frac{a}{1+a} - \frac{b}{1+b}.$$

□

13.2. **The space** $(C(\Omega, X), \rho)$**.** Recall that C_∞ is a metric space with the metric

$$d(z,w) = \begin{cases} \frac{|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}, & z, w \in \mathbb{C} \\ \frac{2}{1+|z^2|} & z \in \mathbb{C}. \end{cases} \tag{7}$$

Given $\Omega \subset \mathbb{C}$ open and X a complete metric space (usually either $\mathbb{C}$ or $\mathbb{C}_\infty$), let $C(\Omega, X)$ denote the space of continuous functions $f : \Omega \to X$. Given a compact set $K \subset \Omega$, the function $d_K : C(\Omega, X) \times C(\Omega, X) \to [0, \infty)$ defined by

$$d_K(f, g) = \sup\{d(f(z), g(z)) : z \in K\}$$

is a semimetric on $C(\Omega, X)$.

Given an exhaustion (K_n) of Ω, let $d_n = d_{K_n}$ and let

$$\rho_n(f, g) = \frac{d_n(f, g)}{1 + d_n(f, g)}.$$

Thus ρ_n is a semimetric by Lemma 13.3. Note, that for each f, g, the series

$$\sum_{n=1}^{\infty} \frac{\rho_n(f, g)}{2^n}$$

converges with value less than 1. Define $\rho(f, g)$ by this series. It of course depends upon the choice of exhaustion.

Proposition 13.4. *ρ is a metric on $C(\Omega, X)$.* †

Proof. Standard arguments show each d_n is a semimetric and thus, by Lemma 13.3, so is each ρ_n and therefore ρ. It remains to show ρ is positive definite; i.e., if $\rho(f, f) = 0$, then $f = 0$. If $\rho(f, f) = 0$, then $d_n(f, f) = 0$ for each n. Hence $f = 0$ on each K_n and since $\Omega = \cup K_n$, it follows that $f = 0$. □

Proposition 13.5. *If $\delta > 0$ and $K \subset \Omega$ is compact, then there exists a $\epsilon > 0$ such that for each $\psi \in C(\Omega, X)$,*

$$\{g : \rho(\psi, g) < \epsilon\} := B_\rho(\psi; \epsilon) \subset B_K(\psi; \delta) := \{g \in C(\Omega, X) : d_K(\psi, g) < \delta\}.$$

In particular, for each $K \subset \Omega$ compact, each $\psi \in C(\Omega, X)$ and each $\delta > 0$, the set $B_K(\psi; \delta)$ is open in $(C(\Omega, X), \rho)$.

Conversely, for each $\epsilon > 0$ there is a compact set K and $\delta > 0$ such that for each ψ,

$$B_K(\psi; \delta) \subset B_\rho(\psi; \epsilon).$$

In particular, if O is open in $(C(\Omega, X))$, then for each $\psi \in O$ there is a compact $K \subset \Omega$ and $\delta > 0$ such that $B_K(\psi;\delta) \subset O$.

Finally, O is open in $(C(\Omega, X))$ if and only if for each $\psi \in O$ there is a compact set K and an $\epsilon > 0$ such that $B_K(\psi;\epsilon) \subset O$. In particular, the open sets in $(C(\Omega, X), \rho)$ do not depend on the choice of exhaustion (K_n). †

Proposition 13.5 says the collection of sets $\{B_K(f;\delta) : K, f, \delta\}$ is a base for the topology of $(C(\Omega, X), \rho)$.

Sketch of proof. Suppose $K \subset \Omega$ compact and $\delta > 0$ are given. There is an N such that $K \subset K_N$ by Lemma 13.1. Now there is a constant C such that $\rho_N(f, g) \leqslant C\rho(f, g)$. Choose $\epsilon > 0$ so that $\mu = C\epsilon < 1$ and $\frac{\mu}{1-\mu} < \delta$. If $\psi \in C(\Omega, X)$ and $\rho(\psi, g) < \epsilon$, then $t = \rho_N(\psi, g) < C\epsilon$ and hence

$$d_N(\psi, g) = \frac{1}{1-t} < \frac{\mu}{1-\mu} < \delta.$$

Thus $\{g : \rho(\psi, g) < \epsilon\} \subset \{g : d_N(\psi, g) < \delta\} \subset B_K(\psi;\delta)$ and hence $B_K(\psi;\delta)$ is open in $(C(\Omega, X), \rho)$. In particular, if $O \subset C(\Omega, X)$ and for each $\psi \in O$ there is a compact set K and $\delta > 0$ such that $B_K(\psi, \delta) \subset O$, then O is open.

Conversely, let $\epsilon > 0$ be given. Choose N such that

$$\sum_{n=N+1} \frac{1}{2^n} < \frac{1}{2}\epsilon.$$

Choose $K = K_N$ and $\delta < \frac{1}{2}\epsilon$. Suppose $\psi \in C(\Omega, X)$ and $d_K(\psi, g) < \delta$, then $\rho_K(\psi, g) < \delta$ too and hence

$$\begin{aligned}\rho(\psi, g) \leqslant& \sum_{n=1}^{N} \frac{\rho_n(\psi, g)}{2^n} + \frac{\epsilon}{2}\\ \leqslant& \rho_N(\psi) \sum_{n=1}^{N} \frac{1}{2^n} + \frac{\epsilon}{2} < \epsilon.\end{aligned}$$

Thus $B_K(\psi, g) \subset B_\rho(\psi;\epsilon)$. □

Proposition 13.6. *A sequence (f_n) in $(C(\Omega, X), \rho)$ converges to an $f \in (C(\Omega), X)$ if and only if for each compact subset $K \subset \Omega$, the sequence $(f_n|_K)$ converges to $f|_K$ uniformly. Similarly, (f_n) is Cauchy in $C(\Omega, X)$ if and only if for each compact $K \subset \Omega$ the sequence $(f_n|_K)$ is Cauchy in $C(K, X)$.* †

The conclusion of Proposition 13.6 is often expressed less formal as: (f_n) *converges uniformly on compact sets* to f.

Proof. Suppose (f_n) converges to f uniformly on compact sets to f. Let $\epsilon > 0$ be given. By Proposition 13.5, there is a compact K and $\delta > 0$ such that $B_K(f;\delta) \subset B_\rho(f;\delta) = \{g : d(f, g) < \rho\}$. Since $(f_n|_K)$ converges uniformly to $f|_K$, there is an N such that $d_K(f_n, f) < \delta$ for $n \geqslant N$. Hence (f_n) converges to f in $(C(\Omega, X), \rho)$.

Now suppose (f_n) converges to f in $(C(\Omega, X), \rho)$ and let $K \subset \Omega$ compact and $\epsilon > 0$ be given. By 13.5, there is a $\delta > 0$ such that $B_\rho(f; \delta) \subset B_K(f; \epsilon)$. Hence, there is an N such that if $n \geqslant N$, then $f_n \in B_K(f; \epsilon)$; i.e., $d_K(f, f_n) < \epsilon$. Thus $(f_n|_K)$ converges uniformly to $f|_K$.

The second part of the proposition is left as an exercise. □

Theorem 13.7. *$C(\Omega, X)$ is complete.*

For K a compact set and X complete, it is well known from advanced calculus that $C(K, X)$ (the continuous X-valued functions on K in the supremum norm is complete).

Sketch of proof. Suppose (f_n) is Cauchy in $C(\Omega, X)$. Since $\{z\} \subset \Omega$ is compact, $(f_n(z))$ is a Cauchy sequence in the complete metric space X and thus converges in X. Hence there is a function $f : \Omega \to X$ such that (f_n) converges to f pointwise. Fix $K \subset \Omega$ compact. It follows that there is a continuous function $f_K : K \to X$ such that $(f_n|_K)$ converges uniformly to f_K. It follows that $f_K = f|_K$ and hence (f_n) converges to f in $C(\Omega, X)$, by Proposition 13.6. □

13.3. **Normal Families.** Recall, that for a metric space Y, compactness and sequential compactness are equivalent. A subset $\mathscr{F}$ of $C(\Omega, X)$ is *normal* if every sequence from $\mathscr{F}$ has a convergent subsequence. Thus $\mathscr{F}$ is normal if and only if $\overline{\mathscr{F}}$ (closure) is (sequentially) compact. In the old days, a set with compact closure was said to be *precompact*.

Recall a subset Z of a metric space Y is totally bounded if for each $\delta > 0$ there exists a positive integer N and points $y_1, \dots, y_N \in Z$ such that $Z \subset \cup B_\rho(y_j; \delta)$; and Z is compact if and only if it is complete and totally bounded.

Proposition 13.8. *A (nonempty) subset $\mathscr{F}$ of $C(\Omega, X)$ is normal if and only if for each $K \subset \Omega$ compact and $\delta > 0$, there exists a positive integer N and $f_1, \dots, f_N \in \mathscr{F}$ such that*

$$\mathscr{F} \subset \cup_{j=1}^{N} B_K(f_j; \delta).$$

†

Since $C(\Omega, X)$ is complete, the proof amounts to showing the inclusion condition is equivalent to total boundedness of $\mathscr{F}$.

Proof. Suppose $\mathscr{F}$ is normal and let K compact and $\delta > 0$ be given. Choose $\epsilon > 0$ as in Proposition 13.5 so that for each ψ the inclusion $\{g : \rho(\psi, g) < \epsilon\} \subset B_K(\psi; \delta)$ holds. By hypothesis $\overline{\mathscr{F}}$ is compact and therefore totally bounded. Thus, there exists $g_1, \dots, g_N \in \overline{\mathscr{F}}$ such that

$$\overline{\mathscr{F}} \subset \cup\{g : \rho(g_j; \frac{\epsilon}{2})$$

Choose f_j such that $f_j \in \mathscr{F}$ and $\rho(f_j, g_j) < \frac{\epsilon}{2}$. Thus,

$$\overline{\mathscr{F}} \subset \cup\{g : \rho(f_j; \epsilon) \subset \cup B_K(f_j; \delta).$$

Conversely, $\mathscr{F}$ has the inclusion property and let $\epsilon > 0$ be given. There is a $\delta > 0$ such that $B_K(\psi;\delta) \subset B_\rho(\psi;\frac{\epsilon}{2})$ independent of $\psi \in C(\Omega, X)$. By hypothesis, there is an N and $f_1, \dots, f_N \in C(\Omega, X)$ such that

$$\mathscr{F} \subset \cup B_K(f_j;\delta).$$

Hence $\overline{\mathscr{F}} \subset \cup B_\rho(f_j;\epsilon)$ and we conclude that $\overline{\mathscr{F}}$ is totally bounded and hence compact. □

13.4. **Arzela-Ascoli.** A subset $\mathscr{F} \subset C(\Omega, \mathbb{C})$ is *equicontinuous* on a subset $E \subset \Omega$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $q \in \Omega$, $p \in E$ and $|q - p| < \delta$ and $f \in \mathscr{F}$, then $|f(q) - f(p)| < \epsilon$. In the case $E = \Omega$, we say $\mathscr{F}$ is equicontinuous; and in the case $E = \{p\}$, we say $\mathscr{F}$ is equicontinuous at p. Finally, $\mathscr{F}$ is *pointwise equicontinuous* if $\mathscr{F}$ is equicontinuous at each point of Ω.

The following lemma is the uniformly over $\mathscr{F}$ version of the statement that a continuous function on a compact set is uniformly continuous.

Lemma 13.9. *If $\mathscr{F} \subset C(\Omega, \mathbb{C})$ is pointwise equicontinuous, then $\mathscr{F}$ is equicontinuous on each compact subset K of Ω.* †

Proof. Let $K \subset \Omega$ compact and $\epsilon > 0$ be given. For each point $w \in K$ there is a δ_w such that if $|z - w| < \delta_w$ (and $z \in \Omega$) and $f \in \mathscr{F}$, then $|f(z) - f(w)| < \frac{\epsilon}{2}$. The collection $\mathscr{O} = \{B(w;\frac{\delta_w}{2}) : w \in K\}$ is an open cover of K. Let δ be the Lebesgue covering number for $\mathscr{O}$. Thus for each $p \in K$ there is a $w \in K$ such that $B(p;\delta) \subset B(w;\frac{\delta_w}{2})$. Now suppose $q \in \Omega$ and $|p - q| < \delta$. It follows that $|q - w| \leqslant |q - p| + |p - w| < \delta_w$. Hence $|p - w|, |w - q| < \delta_w$ and therefore $|f(p) - f(q)| \leqslant |f(p) - f(w)| + |f(w) - f(q)| < \epsilon$. □

Before proceeding further, we recall a standard version of the Arzela-Ascoli theorem from undergraduate analysis. Recall, for K a compact metric space, $C(K, \mathbb{C})$ is the space of continuous functions with the metric $d(f, g) = \max\{|f(x) - g(x)| : x \in K\}$ and that this space is complete. A subset $\mathscr{F} \subset C(K, \mathbb{C})$ is *pointwise bounded* if for each $p \in K$ the set $\{f(p) : f \in \mathscr{F}\} \subset \mathbb{C}$ is bounded.

Lemma 13.10. *A subset F of a metric space Y is precompact if and only if each sequence from F has a convergent subsequence (in Y).* †

Proof. If F is precompact, then $\overline{F}$ is compact and the result follows from the equivalence of compactness and sequential compactness in a metric space.

Conversely, suppose every sequence from F has a convergent subsequence and suppose (f_n) is a sequence from $\overline{F}$. For each n there exists a $g_n \in F$ such that $d(f_n, g_n) < \frac{1}{n}$. The sequence (g_n) has a subsequence (g_{n_k}) that converges to some $g \in Y$. It follows that (f_{n_k}) converges to g too. □

The following version of Arzela-Ascoli is one often encountered in undergraduate analysis.

Theorem 13.11 (Arzela-Ascoli I). *If K is a compact metric space and $\mathscr{F} \subset C(K, \mathbb{C})$ is pointwise bounded and equicontinuous, then every sequence from $\mathscr{F}$ has a subsequence that converges in $C(K, \mathbb{C})$ (that is uniformly). In particular, $\overline{\mathscr{F}}$ is compact in $C(K, \mathbb{C})$.*

Theorem 13.12 (Arzela-Ascoli II). *If $\mathscr{F} \subset C(\Omega, \mathbb{C})$ is pointwise bounded and pointwise equicontinuous, then $\mathscr{F}$ is normal.*

Sketch of proof. By Lemma 13.9, $\mathscr{F}$ is equicontinuous on compact subsets of K. Let (f_m) be a sequence from $\mathscr{F}$. Fix an exhaustion (K_n) of Ω. Note that $\mathscr{F}$ (restricted to K_1) satisfies the hypotheses of Theorem 13.11. Hence there is a subsequence $(f_{1,j})$ of (f_m) such that $(f_{1,j}|_{K_1})$ converges uniformly (on K_1) to some continuous function g_1 on K_1. Likewise, there is a subsequence $(f_{2,j})$ of $(f_{1,j})$ such that $(f_{2,j}|_{K_2})$ converges uniformly (on K_2) to some continuous function g_2 on K_2 with $g_2|_{K_1} = g_1$. Continuing in this fashion, gives sequence $(f_{k,j})$ converging uniformly to a continuous function g_k in K_k (and hence on K_ℓ for each $\ell \leqslant k$) with $g_k|_{K_\ell} = g_\ell$ for $\ell \leqslant k$ and such that $(f_{k+1,j})$ is a subsequence of $(f_{k,j})$. In particular, there is a continuous function $g : \Omega \to \mathbb{C}$ such that $g|_{K_\ell} = g_\ell$ for each ℓ. Let $h_j = (f_{j,j})$. Thus (h_j) is, for each k, eventually a subsequence of $(f_{k,j})$. In particular, (h_j) converges to $g \in C(\Omega, \mathbb{C})$ uniformly on each K_k and hence in the space $C(\Omega, \mathbb{C})$. □

14. The space of analytic functions on Ω

Given an open set $\Omega \subset \mathbb{C}$, let $H(\Omega)$ denote the subspace of $C(\Omega, \mathbb{C})$ consisting of analytic functions. In particular, the algebra $H(\Omega)$ is endowed with the metric it inherits from $C(\Omega, \mathbb{C})$.

Theorem 14.1. *If (f_n) is a sequence from $H(\Omega)$ that converges to an $f \in C(\Omega, \mathbb{C})$, then $f \in H(\Omega)$ and moreover, for each k, the sequence $(f_n^{(k)})$ converges to $f^{(k)}$ (in the metric of $C(\Omega, \mathbb{C})$). In particular, $H(\Omega)$ is complete.*

Proof. Given a triangle T contained in Ω (meaning T and its interior are contained in Ω), there is a compact set K such that $T \subset K^\circ \subset \Omega$. Since (f_n) converges to f uniformly on K, it follows, from Cauchy's Theorem, that

$$0 = \lim_n \int_T f_n = \int_T f.$$

Thus, by Morera's Theorem, f is analytic. In particular, $H(\Omega)$ is a closed subset of the complete metric space $C(\Omega, \mathbb{C})$ and is therefore itself complete.

Fix $y \in \Omega$ and a positive integer k and an $R > 0$ such that $\overline{B(y; 2R)} \subset \Omega$. Let $M_n = \{|f(w) - f(w)| : w \in \overline{B(y; 2R)}$. Since (f_n) converges uniformly to f on compact sets, M_n converges to 0. Given $z \in B(y; R)$, Cauchy's estimate (Corollary 2.2 (iv)) gives

$$|f^{(k)}(z) - f_n^{(k)}(z)| \leqslant \frac{k! M_n}{R^k}$$

Hence $(f_n^{(k)})$ converges uniformly to f on $B(y;R)$. If $K \subset \Omega$ is compact, then (as K is totally bounded) there is an $R > 0$ and points $y_1, \dots y_N$ such that K is covered by the balls $B(y_j; 2R)$. It follows that $(f_n^{(k)})$ converges uniformly to f on K. $\square$

Theorem 14.2 (Hurwitz). *Suppose $\Omega \subset \mathbb{C}$ is an open set, (f_n) is a sequence from $H(\Omega)$ that converges to f (in $H(\Omega)$) and $R > 0$ and $y \in G$. If $\overline{B(y;R)} \subset \Omega$ and f does not vanish on $\{|z-y| = R\}$, then there is an $N \in \mathbb{N}$ such that for all $n \geqslant N$ the functions f_n and f have the same number of zeros in $B(y;R)$.*

In particular, if $\Omega = G$ is a domain and each f_n does not vanish in G, then either f is identically 0, or f does not vanish in G.

Proof. Let $\eta = \{|f(z)| : |z-y| = R\}$. By compactness and the hypotheses, $\eta > 0$. Since (f_n) converges uniformly to f on compact sets, there is an N such that for $n \geqslant N$ and $|z-y| = R$,

$$|f_n(z) - f(z)| < \frac{\eta}{2} < |f(z)|.$$

Thus, by Theorem 11.3, (f_n) and f have the same number of zeros in $B(y;R)$.

Now suppose the f_n never vanish, $\Omega = G$ is a domain and f is not identically 0. Because G is a connected, it follows that, given $y \in G$, there is an $R > 0$ such that $B(y;R) \subset G$ and f does not vanish on $\{|z-y| = R\}$. Hence, by what has already been proved, f has no zeros in $B(y;R)$ and in particular, $f(y) \neq 0$. Hence f has no zeros. $\square$

14.1. **Montel's Theorem.** A set $\mathscr{F} \subset C(\Omega, \mathbb{C})$ is bounded on a subset U of Ω if the set $\{|f(z)| : z \in U,\ f \in \mathscr{F}\}$ is bounded. It is *locally bounded* if for each $y \in \Omega$ there is an open set $y \in U$ such that $\mathscr{F}$ is bounded on U.

Lemma 14.3. *If $\mathscr{F}$ is locally bounded, then $\mathscr{F}$ is bounded on each compact $K \subset \Omega$.* †

Theorem 14.4 (Montel's Theorem). *A subset $\mathscr{F} \subset H(\Omega)$ is normal if and only if it is locally bounded.*

Proof. Arguing by contradiction, suppose $\mathscr{F}$ is normal, but not locally bounded. In this case there is an $y \in \Omega$ and a bounded open set $y \in U$ such that $\overline{U} \subset \Omega$ such that

$$\sup\{|f(z)| : z \in U,\ f \in \mathscr{F}\} = \infty.$$

Thus there exists, for each $n \in \mathbb{N}^+$, a point $z_n \in U$ and $f_n \in \mathscr{F}$ such that $|f_n(z_n)| \geqslant n$. Since $\mathscr{F}$ is normal, there is a subsequence (f_{n_k}) of (f_n) that converges to some $f \in H(\Omega)$. In particular, (f_{n_k}) converges to f uniformly on $\overline{U}$. Since also f is bounded on $\overline{U}$, it follows that $(f_{n_k}(z_{n_k}))_k$ is a bounded sequence, a contradiction.

Now suppose $\mathscr{F}$ is locally bounded. The plan is to show that $\mathscr{F}$ is pointwise equicontinuous and apply Arzela-Ascoli (Theorem 13.12). Accordingly, fix $y \in \Omega$. By the local bounded hypothesis, there exist $r, M > 0$ such that $\overline{B(y;r)} \subset \Omega$ and, for all $z \in \overline{B(y;r)}$ and $f \in \mathscr{F}$,

$$|f(z)| \leqslant M.$$

Let $\gamma(s) = y + r\exp(2i\pi s)$ $(0 \leqslant s \leqslant 1)$. For $z \in B(y; \frac{r}{2})$ and $f \in \mathscr{F}$, Cauchy's formula gives,

$$\begin{aligned} |f(z) - f(y)| \leqslant & \int_0^1 \frac{|f(\gamma(s))(z-y)|}{|\gamma(s) - y|\,|\gamma(s) - z|} r\,ds \\ \leqslant & |z-y| \int_0^1 \frac{Mr}{(\frac{r}{2})^2} = |z-y|\frac{4M}{r} \end{aligned}$$

Hence $\mathscr{F}$ is equicontinuous at y and the proof is complete. □

Corollary 14.5. *A subset $\mathscr{F} \subset H(\Omega)$ is compact if and only if it is closed and locally bounded.* †

15. The space of meromorphic functions on Ω

Recall the metric d on C_∞ from equation (7). The topology induced by this metric is one point compactification of $\mathbb{C}$; i.e., neighborhoods of ∞ are complements of compact subsets of $\mathbb{C}$.

Proposition 15.1. *A subset $O \subset \mathbb{C}_\infty$ is open if and only if $O\backslash\{\infty\}$ is open in $\mathbb{C}$ and, if $\infty \in O$, then there exists a compact set $K \subset \mathbb{C}$ such that $\tilde{K} \subset O$.* †

Suppose $\Omega \subset \mathbb{C}$ is open. Given f meromorphic on Ω, define $f(p) = \infty$ if p is a pole of f. In this way, f determines a function (still denoted by f) in $C(\Omega, \mathbb{C}_\infty)$ (by the definition of pole and Proposition 15.1). Let $M(\Omega) \subset C(\Omega, \mathbb{C}_\infty)$ denote the set of meromorphic functions on Ω. Observe, If $g \in M(\Omega)$, then $\frac{1}{g} \in M(\Omega)$ too.

A key feature of the metric d is

$$d(z,w) = \begin{cases} d(\frac{1}{z}, \frac{1}{w}) & z, w \in \mathbb{C} \\ d(\frac{1}{z}, \infty) & z \in \mathbb{C}, w = 0. \end{cases}$$

with the following lemma as an immediate consequence,

Lemma 15.2. *Suppose $G \subset \mathbb{C}$ is a domains. If $(f_n) \in H(\Omega)$ converges to $f \in M(\Omega)$, then $(\frac{1}{f_n})$ converges to $\frac{1}{f}$ in $M(\Omega)$.* †

Proposition 15.3. *If (f_n) is a sequence from $M(G) \subset C(G, \mathbb{C}_\infty)$ (resp. $H(G) \subset C(G, \mathbb{C}_\infty)$) and if (f_n) converges to f in $C(G, \mathbb{C}_\infty)$, then either f is meromorphic (resp. $f \in H(G)$) or f is identically equal to ∞.* †

Proof. Fix $y \in G$. First suppose $f(y) \neq \infty$. Since the set $U = \{w : |f(y) - w| < 1\}$ is open in $\mathbb{C}$, there is a $\delta > 0$ such that $V = \{w : d(w, f(y)) < \delta\} \subset U$ by Proposition 15.1. Choose $r > 0$ such that $B = \overline{B(y;r)} \subset G$ and so that $d(f(z), f(y)) < \frac{\rho}{2}$ for $z \in B$. Since (f_n) converges to f uniformly on B, there is an N such that if $n \geqslant N$ and $z \in B$, then $d(f_n(z), f(z)) < \frac{\rho}{2}$. It follows that, for $n \geqslant N$ and $z \in B$,

$$d(f_n(z), f(y)) \leqslant d(f_n(z), f(z)) + d(f(z), f(y)) < \rho$$

and therefore $|f_n(z)| \leqslant |f(y)| + 1$. Hence, for $n \geqslant N$, the function f_n is analytic in $B(y;r)$. By Theorem 14.1 (applied to $G = B(y;r)$, f is analytic in a neighborhood of y.

Now suppose $f(y) = \infty$. Using Lemma 15.2, the argument of the previous paragraph shows $\frac{1}{f}$ is analytic in a neighborhood U of y. If f is not identically ∞, then, since G is connected, f is not identically ∞ on U. It follows that the zeros of $\frac{1}{f}$ in U are isolated and hence f is meromorphic in U.

In the case each (f_n) is analytic, the functions $\frac{1}{f_n}$ have no zeros. Hence, by Theorem 14.2 (Hurwitz), $\frac{1}{f}$ is either identically zero or not zero in a neighborhood of y. □

Corollary 15.4. *The subspace $M(\Omega) \cup \{\infty\}$ of $C(\Omega, \mathbb{C}_\infty)$ is complete.* †

16. The Riemann mapping theorem

Recall, from Corollary 7.10 item (v) that, if G is simply connected, then every nowhere vanishing analytic function on G has an analytic square root.

Theorem 16.1 (Riemann mapping)**.** *If $G \subset \mathbb{C}$ is a domain (open and connected), every nowhere vanishing function on G has a square root and $G \neq \mathbb{C}$ and $y \in G$, then there is an analytic bijection $f : G \to \mathbb{D}$. Further, given $y \in G$, there is a unique analytic bijection $f : G \to \mathbb{D}$ with $f(y) = 0$ and $f'(y) > 0$.*

The proof of uniqueness is a consequence of Theorem 12.5 (Schwarz's lemma). The details are left as an exercise.

Lemma 16.2. *There exists a one-one analytic function $f : G \to \mathbb{D}$ such that $f(y) = 0$ and $f'(y) > 0$.* †

Proof. Choose a point $w \notin G$. Since $h(z) = z - w$ does not vanish in G, there is an analytic function g on G such that $g^2 = h = z - w$. If $g(\zeta) = g(z)$, then $\zeta - w = z - w$ and hence $\zeta = w$. Thus g is one-one.

Fix $p \in g(G)$. There is a $\zeta \in G$ such that $g(\zeta) = p$. If $g(z) = -p$, then $\zeta - w = p^2 = z - w$ and consequently $\zeta = w$, a contradiction. Theorem 8.4, $g(G)$ is open. Hence there is an $r > 0$ such that $B(p;r) \subset g(G)$ and therefore $B(-p;r) \subset \mathbb{C} \backslash g(G)$. This latter inclusion implies $|g(z) - (-p)| \geqslant r$ for $z \in G$ and therefore $|\frac{r}{g(z)+p}| < 1$. Hence

$$F(z) = \frac{r}{2(g(z) + p)}$$

defines a one-one analytic function on G with values in $\mathbb{D}$. Since F is one-one, its derivative never vanishes by Proposition 8.3. Thus, post composition with an appropriate mobius mapping $\varphi_b = \frac{z-b}{1-\bar{b}z}$ $(b = F(y))$ followed by a rotation produces the desired f. □

Proof of Theorem 16.1. Let

$$\mathscr{F} = \{f : G \to \mathbb{D} : f \text{ is one-one },\ f(y) = 0,\ f'(y) > 0\}.$$

By Lemma 16.2, $\mathscr{F}$ is not empty. By construction, $\mathscr{F}$ is locally bounded. Hence, by Theorem 14.4 (Montel), $\mathscr{F}$ has compact closure in $C(G, \mathbb{C})$. Suppose (f_n) is a sequence from $\mathscr{F}$ that converges to some f. Since each f_n' does not vanish (by Proposition 8.3), either f' is identically 0 or f' is never 0 by Theorem 14.2 (Hurwitz). In the former case, f is constant and thus constantly equal to 0 and is in $\mathscr{F}$. In the latter case, $f(y) = 0$ and $f'(y) \neq 0$ and thus $f'(y) > 0$. Moreover, since each f_n is one-one and f is not constant, another application of Theorem 14.2 (Hurwitz) implies f is one-one. Finally, it is evident that f maps into $\overline{\mathbb{D}}$. On the other hand, since f is one-one, the open mapping theorem implies $f(G)$ is open and thus a subset of $\mathbb{D}$. We conclude in any case $f \in \mathscr{F} \cup \{0\}$. Hence $\mathscr{F} \cup \{0\}$ is the closure of $\mathscr{F}$ and is therefore compact in $C(G, \mathbb{C})$.

The mapping $\mathscr{F} \cup \{0\} \ni f \to f'(y) \in \mathbb{C}$ is continuous by Theorem 14.1 and takes values in $[0, \infty)$. Since $\mathscr{F} \cup \{0\}$ is compact, this map attains its maximum M. Let $f \in \mathscr{F}$ be any such that $f'(y) = M$. Suppose, by way of contradiction, there is a $w \in \mathbb{D}$ not in the range of f. Consider

$$\psi(z) = \varphi_w(f(z)) = \frac{f(z) - w}{1 - \bar{w} f(z)}.$$

Since φ_w doesn't vanish and G is simply connected, there is an analytic function $h : G \to \mathbb{C}$ such that $h^2 = \psi$. Let

$$g = \lambda \varphi_{h(y)} \circ h = \lambda \frac{h(z) - h(y)}{1 - \bar{h(y)} h(z)},$$

where λ is unimodular and chosen so that $g'(y) > 0$. Compute,

$$g'(y) = \frac{|h'(y)|}{1 - |h(y)|^2}$$

and

$$|h'(y)| = \frac{f'(y)(1 - |w|^2)}{2\sqrt{|w|}}.$$

Combining these last two equations gives the contradiction

$$g'(y) = \frac{1 + |w|}{2\sqrt{|w|}} f'(y) > f'(y),$$

and completes the proof of the theorem. □

Corollary 16.3. *Suppose $G \subset \mathbb{C}$ is open. The following are equivalent.*

(i) $G \neq C$ and if $f : G \to \mathbb{C}$ is analytic and $\gamma : [0, 1] \to G$ is a closed rectifiable path, then

$$\int_\gamma f \, dz = 0.$$

(ii) $G \neq \mathbb{C}$ and if $f : G \to \mathbb{C}$ is analytic and if $\gamma, \delta : [0, 1] \to G$ are rectifiable paths such that $\gamma(0) = \delta(0)$ and $\gamma(1) = \delta(1)$, then

$$\int_\gamma f = \int_\delta f.$$

(iii) if $G \neq \mathbb{C}$ and $f : G \to \mathbb{C}$, then f has a primitive.

(iv) If $G \neq \mathbb{C}$ and $f : G \to \mathbb{C}$ is analytic and never vanishes, then there exists an analytic function $g : G \to \mathbb{C}$ such that $f = e^g$.
(v) If $G \neq \mathbb{C}$ and $f : G \to \mathbb{C}$ is analytic and never vanishes, then f has a square root in G.
(vi) There is an analytic bijection $f : G \to \mathbb{D}$.

†

Sketch of proof. It is easy to show, if $G \subset \mathbb{C}$ and if there is an analytic bijection $f : G \to \mathbb{D}$, then $G \neq \mathbb{C}$ and G is simply connected; i.e., item (vi) implies item (i). On the other hand, the proof of Corollary 7.10 showed (without the assumption $G \neq \mathbb{C}$) that item (i) implies item (ii) implies item (iii) implies item (iv) implies item (v). Theorem 16.1 shows item (v) implies item (vi). □

17. Factorization of analytic functions

17.1. **Infinite products.** Given a sequence (z_n) from $\mathbb{C}$, let $\prod_{j=1}^{\infty} z_j$ denote the sequence $p_n = \prod_{j=1}^{n} z_j$ and also the limit of this sequence, called the *infinite product*, if it exists.

Lemma 17.1. *If none of the z_n are zero and if the infinite product exists (converges) and is not zero, then (z_n) converges to 1. On the other hand, with $|y| < 1$ fixed and $z_n = y^n$, the product $\prod z_n$ and (z_n) both converge to 0.* †

Proof. Let $0 \neq p = \lim p_n$. Since none of the z_n are zero, none of p_n are 0. Hence $z_{n+1} = \frac{p_{n+1}}{p_n}$ and it follows that (z_n) converges to 1. □

Let $\Phi(z) = \log(|z|) + i\theta$, where $-\pi \leqslant \theta \leqslant \pi$. In particular, $\exp(\Phi(z)) = z$.

Proposition 17.2. *The infinite product $\prod z_n$ converges to a non-zero number if and only if the series $\sum \Phi(z_n)$ converges.* †

Proof. Continue to let $p_n = \prod_{j=1}^{n} z_j$ and let $s_n = \sum_{j=1}^{n} \Phi(z_n)$. In particular, $\exp(s_n) = p_n$. Hence, if (s_n) converges to s, then $p_n = \exp(s_n)$ converges to $\exp(s) \neq 0$.

Now suppose (p_n) converges to $p \neq 0$. Let φ denote a branch of the logarithm that is continuous at p. In particular, there is an N such that, for each $n \geqslant N$, the point p_n is in the domain of φ and $(\varphi(p_n))_{n\geqslant N}$ converges to $\varphi(p)$. From here on we take $n \geqslant N$. Since $\exp(s_n) = p_n$, it follows that there exists integers m_n such that $s_n = \varphi(p_n) + 2m_n\pi i$. On the other hand, since $s_{n+1} - s_n = \Phi(z_n)$ and $(\Phi(z_n))$ converges to 1 by Lemma 17.1, it follows that there is a $K \in \mathbb{N}$ and an integer k such that if $n \geqslant K$, then $m_n = k$. Hence (s_n) converges to $\varphi(p) + 2\pi ik$. □

Remark 17.3. If $\prod z_n$ converges to a non-zero number, then $\Re(z_n) > 0$ for n sufficiently large. Hence, eventually, $\Phi(z_n) = \log(z_n)$ (the principal branch). ◇

Given a sequence of non-zero complex numbers, the infinite product $\prod z_n$ *converges absolutely* if the series $\sum \Phi(z_n)$ does.

Proposition 17.4. *If the product $\prod z_n$ converges absolutely, then the product itself converges to a non-zero number p.* †

Proof. If $\prod z_n$ converges absolutely, then $\sum \Phi(z_n)$ converges absolutely (by definition). Hence $\sum \Phi(z_n)$ and by Proposition 17.2, the infinite product converges. □

Lemma 17.5. *Let $\varphi(z) = \log(1+z)$ (with domain $\Re(z) > -1$). If $|z| < \frac{1}{2}$, then*

$$\frac{1}{2}|z| \leqslant |\varphi(z)| \leqslant 2|z|.$$

†

Proof. Let log denote the principal branch. The function $\varphi(z) = \log(1+z)$ has a power series expansion convergent in $B(0;1)$,

$$\varphi(z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}.$$

It follows that, for $|z| \leqslant \frac{1}{2}$,

$$|z - \varphi(z)| \leqslant \sum_{n=1}^{\infty} |z|^n = \frac{|z|}{1-|z|} \leqslant 2|z|.$$

Likewise for $|z| \leqslant \frac{1}{2}$,

$$|z - \varphi(z)| \leqslant \sum_{n=2}^{\infty} |z|^n = \frac{|z|^2}{1-|z|} \leqslant \frac{1}{2}|z|.$$

Hence, for $|z| \leqslant \frac{1}{2}$,

$$|\varphi(z)| \geqslant |z| - |z - \varphi(z)| \geqslant \frac{1}{2}|z|.$$

□

Lemma 17.6. *Suppose (z_n) is a sequence of complex numbers none of which are 1. The series $\sum \Phi(1+z_n)$ converges absolutely if and only if $\sum z_n$ converges absolutely.* †

Proof. Suppose $\sum z_n$ converges absolutely. In this case there is an N such that $|z_n| \leqslant \frac{1}{2}$ for $n \geqslant N$. For these n, $\Phi(z) = \log(z)$. Hence, by Lemma 17.5, $\sum |\Phi(1+z_n)|$ converges by comparison to $\sum |z_n|$.

Conversely, if $\sum |\Phi(1+z_n)|$, then there is an N such that if $n \geqslant N$, then $|z_n| < \frac{1}{2}$. Again $\Phi(z) = \log(z)$ for these n. Hence, the other inequality of Lemma 17.5 and the comparison test imply $\sum |z_n|$ converges. □

Proposition 17.7. *Suppose (z_n) is a sequence of non-zero numbers. The product $\prod z_n$ converges absolutely if and only if the series $\sum (z_n - 1)$ does.* †

Lemma 17.8. *Let (X,d) be a compact metric space and suppose (g_n) is a sequence of continuous $\mathbb{C}$-valued functions on X. If the series $\sum |g_n|$ converges uniformly, then*

(a) the product $\prod (1+g_n)$ converges uniformly to some f;

(b) the product $\prod(1+g_n)$ *converges pointwise absolutely;*
(c) there is an N *such that* $f(x) = 0$ *if and only if there is an* $n \leqslant N$ *such that* $g_n(x) = -1$.

†

Proof. Pointwise absolute convergence follows from combining Lemma 17.6 and Proposition 17.7.

Since $\sum |g_n|$ converges uniformly, there is an N such that if $n \geqslant N$, then $|g_n| < \frac{1}{2}$. By Lemma 17.5, for $n \geqslant N$,

$$|\log(1+g_n)| \leqslant 2|g_n|.$$

By comparison, $\sum_{n=N}^{\infty} \Phi(1+g_n)$ converges uniformly.

Let $s_n = \sum_{j=N}^{n} z_j$ and let $t_n = \sum_{j=N}^{n} \Phi(1+g_n)$ and let s and t denote their uniform limits respectively. Since (s_n) is a sequence of continuous functions converging uniformly, its limit s is continuous. Since X is compact the sequence (s_n) is uniformly bounded. Hence the sequence $t_n = \sum_{j=N}^{n} \Phi(1+g_n)$ is also uniformly bounded; i.e., there is a compact set $K \subset \mathbb{C}$ such that, for each $n \geqslant N$, the range of t_n lies in K. Since the exponential function is continuous on K, it is uniformly continuous on K. It follows that $\exp(t_n)$ converges to $\exp(t)$ uniformly and thus the full product

$$f = \prod(1+g_n) = \prod_{j=1}^{N-1}(1+g_j) \prod_{j=N}^{\infty}(1+g_j) = \prod_{j=1}^{N-1}(1+g_j)\, \exp(t)$$

converges uniformly too. Moreover, if $f(z) = 0$, then there is an $n < N$ such that $1 + g_n(z) = 0$. □

Proposition 17.9. *Suppose* $\Omega \subset \mathbb{C}$ *is open and* (f_n) *is a sequence from* $H(\Omega)$. *If* $\sum_{n=1}^{\infty} |f_n - 1|$ *converges uniformly on compact subsets of* Ω, *then* $f = \prod f_n$ *converges in* $H(\Omega)$ *and pointwise absolutely. Moreover, if* Ω *a domain and if none of the* f_n *are identically zero, then* f *is not identically zero and in this case if* $f(y) = 0$ *then the multiplicity of this zero is the sum of the multiplicity of the zeros of the* f_n *at* y. †

Proof. By Lemma 17.8, $\prod f_n$ converges uniformly on compact sets of Ω and hence, by Theorem 14.1, this product f is in $H(\Omega)$. Lemma 17.8 also implies, given a compact subset $K \subset \Omega$, that there exists an N such that if $z \in K$ and $f(z) = 0$, then there is an $n \leqslant N$ such that $f_n(z) = 0$. Hence, if none of the f_n are identically zero, then the zero sets of f and $F = \prod_{n=1}^{N} f_n$ in K are the same. In particular, for $n > N$ each f_n is never 0 on K and of course f is not identically zero on K. Now suppose Ω is connected and $f(y) = 0$. Choose $r > 0$ such that $K = \overline{B(y;r)} \subset \Omega$. Thus f is not identically zero on K and hence is not identically zero on Ω. Moreover, since f and F have the same zero sets, the conclusion about the multiplicity of the zero of f at y follows. □

17.2. **Weierstrass Factorization.** Let $E_0 = 1 - z$ and, for $p \in \mathbb{N}^+$, let

$$E_p = (1-z)\, \exp(\sum_{j=1}^{p} \frac{z^j}{j}).$$

These functions are the *Weierstrass elementary factors*. They have a simple zero at 1 and no other zeros.

Lemma 17.10. *If* $|z| \leqslant 1$, *then* $|1 - E_p(z)| \leqslant |z|^{p+1}$. †

Proof. The case $p = 0$ is evident. Accordingly, suppose $p \geqslant 1$. Observe

$$E'(z) = -z^p \exp(\sum_{j=1}^{p} \frac{z^j}{j}) = -\sum_{j=p}^{\infty} b_j z^j$$

where $b_j \geqslant 0$ and the power series has infinite radius of convergence. Since $E_p(0) = 1$, the elementary factor E_p has a power series

$$E_p(z) = 1 - \sum_{j=p+1}^{\infty} a_j z^j$$

with $a_j \geqslant 0$ for all j. Since $E(1) = 0$, we find $\sum a_j = 1$. Thus, for $|z| \leqslant 1$,

$$|E_p(z) - 1| \leqslant |z|^{p+1} \sum_{j=p+1}^{\infty} a_j = |z|^{p+1}.$$

□

Proposition 17.11. *Suppose* (a_n) *is a sequence from* $\mathbb{C}\backslash\{0\}$ *and* (p_n) *is a sequence from* $\mathbb{N}$. *If*

(i) $a_n \neq 0$ *for each* n;
(ii) $\lim |a_n| = \infty$; *and*
(iii)

$$\sum_{n=1}^{\infty} \left(\frac{|r|}{|a_n|}\right)^{p_n+1} \tag{8}$$

converges for all $r \in \mathbb{C}$,

then

(a)

$$f(z) = \prod E_{p_n}(\frac{z}{a_n})$$

converges in $H(\mathbb{C})$ *(uniformly on compact sets) and pointwise absolutely;*
(b) *the zeros of* f *are exactly the* (a_n) *and each zero* a *occurs with multiplicity equal to the number of times* $a = a_n$;
(c) *if* $p_n = n - 1$, *then the series of equation* (8) *does converge for all* $r \in \mathbb{R}$.

In particular, if $\lim |a_n| = \infty$, *then there is an entire function with zeros exactly* a_n *(counted with multiplicity).* †

Proof. Fix $r > 0$ and choose an N so that $|a_n| \geqslant r$ for $n \geqslant N$. If $|z| \leqslant r$ and $n \geqslant N$, then $|\frac{z}{a_n}| \leqslant 1$ and hence, by Lemma 17.10,

$$|E_{p_n}(z) - 1| \leqslant |\frac{z}{a_n}|^{p_n+1} \leqslant |\frac{r}{a_n}|^{p_n+1}.$$

Hence $\sum |E_{p_n}(z) - 1|$ converges uniformly on $\overline{B(0;r)}$ and therefore uniformly on compacts subsets of $\mathbb{C}$. Hence, by Proposition 17.9 the infinite product $\prod E_{p_n}(\frac{z}{a_n})$ converges in $H(\mathbb{C})$ and its zeros are precisely the zeros of the factors (counted with multiplicity) and hence the a_n (counted with multiplicity). □

Theorem 17.12 (Weierstrass factorization)**.** *If $f : \mathbb{C} \to \mathbb{C}$ is entire with zeros (a_n) and if (p_n) is a sequence from $\mathbb{N}$ such that*

$$\sum_{n=1}^{\infty} |\frac{r}{a_n}|^{p_n+1}$$

converges for all $r \in \mathbb{C}$, then there exists an $m \in \mathbb{N}$ and an entire function g such that

$$f(z) = z^m \exp(g(z)) \prod E_{p_n}(\frac{z}{a_n}),$$

with the product converging in $H(\mathbb{C})$ and pointwise absolutely. Moreover such a sequence (p_n) does exist.

Proof. By Proposition 17.11, there is an $m \in \mathbb{N}$ such that f and

$$h(z) = z^m \prod E_{p_n}(\frac{z}{a_n})$$

have precisely the same zero set counting according to multiplicity. Hence the ratio $\frac{f}{h}$ defines an entire function with no zeros. Since $\mathbb{C}$ is simply connected, by Corollary 7.10(iv) there is an entire function g such that $\frac{f}{h} = \exp(g)$. □

Proposition 17.13. *Suppose $\Omega \subset \mathbb{C}$ is open. If (a_n) is a sequence from G with no limit points in G, then there is an analytic function $f : G \to \mathbb{C}$ whose zeros are precisely the a_n counted with multiplicity.* †

Corollary 17.14. *If $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$ is meromorphic, then there exists analytic functions $g, h : \Omega \to \mathbb{C}$ such that $f = \frac{g}{h}$.* †

Proof. Let (p_n) denote the poles of f counted according to their orders. Since the poles of a meromorphic function do not have a limit point in Ω, by Proposition 17.13 there exists an analytic function $h : \Omega \to C$ with zeros precisely (p_n). It follows that $g = fh$ is analytic. □

17.3. **Factorization of sine.**

Proposition 17.15. *Suppose $\Omega \subset \mathbb{C}$ is open and (f_n) is a sequence from $H(\Omega)$. If $\prod f_n$ converges in $H(\Omega)$ to f and if $z \in \Omega$ and $f(z) \neq 0$, then*

$$\frac{f'}{f}(z) = \sum_{n=1}^{\infty} \frac{f_n'}{f_n'}(z)$$

pointwise. †

Actually the convergence is uniform over compact sets where f doesn't vanish (that is, if K is compact and f doesn't vanish on K the the series converges uniformly on K), but this stronger conclusion is not needed for what follows.

Proposition 17.16. *For $z \in \mathbb{C}\backslash\mathbb{Z}$,*

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}.$$

†

Sketch of proof. For positive integers n, let γ_n denote the rectangular path, oriented counterclockwise, connecting the points $\pm(n+\frac{1}{2}) \pm in$. Fix a $y \in \mathbb{C}\backslash\mathbb{Z}$ and consider the meromorphic function

$$F(z) = \frac{\cot(\pi z)}{z^2-y^2}.$$

It has poles at integers k and also at $\pm y$. The residue of $\cot(\pi z)$ at $k \in \mathbb{Z}$ is $\frac{1}{\pi}$ and hence the residue of F at k is $\frac{1}{\pi(k^2-y^2)}$. The residues of F at $\pm y$ are $\pm\frac{\cot(\pm\pi y)}{2y}$. Hence, by the residue theorem,

$$\begin{aligned}\int_{\gamma_n} F_n =& 2\pi i\left[\frac{\cot(\pi y)}{2y} - \frac{\cot(-\pi y)}{2y} - \frac{1}{\pi y^2} + 2\sum_{k=1}^{n}\frac{1}{\pi(k^2-y^2)}\right]\\ =& \frac{i}{y}\left[\pi\cot(\pi y) - \left(\frac{1}{y} + 2\sum_{k=1}^{n}\frac{1}{y^2-k^2}\right)\right].\end{aligned}$$

On the other hand,

$$\lim_{n\to\infty}\int_{\gamma_n} F = 0$$

since $\cot\pi z$ is bounded uniformly on $\cup_n\{\gamma_n\}$ and $\frac{1}{z^2-y^2}$ behaves like $\frac{1}{z^2}$ for $|z|$ large. Thus,

$$\pi\cot(\pi z) = \frac{1}{z} + 2\sum_{k=1}^{\infty}\frac{1}{z^2-k^2}.$$

□

Proposition 17.17. $\sin(\pi z) = \pi z\prod_{j=1}^{\infty}(1-\frac{z^2}{k^2})$. †

Proof. The zero set of $f(z) = \sin(\pi z) = \frac{1}{2i}(\exp(i\pi z) - \exp(-i\pi z))$ is precisely the set $\mathbb{Z}$ (each with multiplicity 1). Since

$$\sum_{n=1}^{\infty}\left|\frac{r}{n}\right|^2$$

converges for each r, we may choose $p_n = 1$ in Theorem 17.12 (Weierstrass factorization) and conclude there is an entire function g such that, using the absolute convergence to rearrange the product,

$$\begin{aligned} f(z) =& z\, \exp(g) \prod_{n\neq 0} E_1(\frac{z}{n}) \\ =& z\, \exp(g) \prod_{n\neq 0} (1-\frac{z}{n}). \end{aligned}$$

For z not an integer (and thus not a zero of f), Proposition 17.15 gives

$$\pi \cot(\pi z) = \frac{f'}{f}(z) = \frac{1}{z} + g'(z) + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}.$$

Comparing with Proposition 17.16 shows $g'=0$ and hence $g=c\in\mathbb{C}$. Rearranging

$$\frac{\sin(\pi z)}{\pi z} = \frac{e^c}{\pi} \prod_{n=1}^{\infty} (1-\frac{z^2}{n^2}).$$

Letting z tend to 0 gives

$$1 = \frac{e^c}{\pi}.$$

□

18. Runge's Theorem

The proof of Runge's theorem presented follows the rather elementary argument of Sarason.

Proposition 18.1. *If $K\subset\Omega\subset\mathbb{C}$ where K is compact and Ω is open, then there exists an N and a path $\Gamma=\sum_{j=1}^N \Gamma_j$ where each Γ_j is a closed curve made of line segments each parallel to either the real or imaginary axis such that*

(i) $n_\Gamma(z)=1$ for $z\in K$;
(ii) $n_\Gamma(z)=0$ for $z\notin\Omega$;
(iii) if $f:\Omega\to\mathbb{C}$ is analytic and $w\in K$, then Cauchy's integral formula holds; i.e.,

$$f(w) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-w}\,dz$$

†

Sketch of proof. Choose $\delta>0$ strictly less than the distance from K to $\partial\Omega$. For $k,\ell\in\mathbb{Z}$, let $R_{k,\ell}$ denote the rectangle in $\mathbb{C}=\mathbb{R}^2$,

$$\mathbb{R}_{k,\ell} = [\delta k, \delta(k+1)] \times [\delta\ell, \delta(\ell+1)].$$

Reusing notation slightly, let $\mathscr{R}=\{R_1,\dots,R_g\}$ enumerate those rectangles $R_{j,k}$ that intersect K. By the choice of δ, we have $R_j\subset\Omega$ for each j. Let σ_j denote the boundary of R_j as a counterclockwise oriented closed path and let $\sigma=\sum\sigma_j$. Thus σ is a (sum of) closed path(s) with the property that for each $z\in K\backslash\{\sigma\}$,

$$n_\sigma(z)=1$$

and for $z \notin \Omega$,

$$n_\sigma(z) = 0.$$

Let $\mathscr{S}$ denote the sides (considered for the moment without orientation) of rectangles from $\mathscr{R}$ that lie on only one rectangle from $\mathscr{R}$ viewed as a path with the orientation inherited from the rectangle in $\mathscr{R}$ it lies in. The *key property* of $\mathscr{S}$ is: *for each vertex $v = (k\delta, \ell\delta)$ the number of sides from $\mathscr{S}$ terminating at v equals the number of sides from $\mathscr{S}$ terminating at p. This number is either* 0, 1 *or* 2. Further, any side of a rectangle in $\mathscr{R}$ that is not in $\mathscr{S}$ is a side of exactly one other rectangle from $\mathscr{R}$, but with the opposite orientation. Hence, letting Γ denote the oriented path built from $\mathscr{S}$ (we don't yet know Γ is closed), if $z \in K\backslash\{\sigma\}$ or if $z \notin \Omega$, then

$$\frac{1}{2\pi i}\int_\Gamma \frac{1}{w-z}\,dw = \frac{1}{2\pi i}\int_\sigma \frac{1}{z-w}\,dw = n_\sigma(z).$$

By continuity,

$$\frac{1}{2\pi i}\int_\Gamma \frac{1}{w-z}\,dw = \begin{cases} 1 \text{ if } z \in K \\ 0 \text{ if } z \notin \Omega. \end{cases}$$

Thus it remains only to show that Γ is a closed path.

Let $(S_1, \dots, S_p)$ be a chain from $\mathscr{S}$; i.e., the terminal point of S_j is the initial point of S_{j+1}. By the critical property if this chain is maximal, then $\sum_{j=1}^p S_j$ is a closed path. Accordingly, suppose $(S_1, \dots, S_p)$ is maximal. The collection $\mathscr{S}\backslash\{S_1, \dots, S_p\}$ also has the key property. Thus an induction argument shows that the sides from $\mathscr{S}$ can be arranged as $\Gamma = \sum \Gamma_j$, where the Γ_j are closed paths. □

Lemma 18.2. *Let $L \subset \mathbb{C}$ be a (closed) line segment of length ℓ (thought of as a path) with midpoint p. If f is continuous on L, then F defined on $\mathbb{C}\backslash L$ by*

$$F(w) = \frac{1}{2\pi i}\int_L \frac{f(z)}{z-w}\,dz$$

is analytic on $A = \{|w-p| > \ell\}$ and hence has a convergent Laurent series expansion,

$$F(z) = \sum_{n=1}^\infty a_n (z-p)^{-n},$$

valid for $|z-p| > \frac{\ell}{2}$ with uniform convergence on compact sets. †

Proof. Analyticity of F follows from Proposition 5.2. It is possible to prove this result by appeal to Proposition 9.4, but a simple direct proof (and essentially half of the proof of Proposition9.4) results from considering the function $G : \{|\zeta| < \frac{2}{\ell}\}\backslash\{0\}$ defined by

$$G(\zeta) = F(\frac{1}{\zeta} + p).$$

Observe that G has a removable singularity at 0 because F vanishes at ∞, and after removing this singularity, $G(0) = 0$. Hence, G has a power series expansion,

$$G(\zeta) = \sum_{j=1}^\infty a_j \zeta^j,$$

valid for $|z| < \frac{2}{\ell}$. Using $F(w) = G(\frac{1}{w-p})$ completes the proof. □

Theorem 18.3 (Runge). *Suppose $K \subset \Omega \subset \mathbb{C}$ where K is compact and Ω is open. If $f : \Omega \to \mathbb{C}$ is analytic, then for each $\epsilon > 0$ there is a rational function r with poles off K such that*

$$\epsilon > \|f - r\|_K := \max\{|f(z) - r(z)| : z \in K\};$$

i.e., f is the uniformly approximable on K by rational functions with poles off K.

Proof. Let Γ be a closed path as in Proposition 18.1. Let δ denote the distance from K to Γ. Write $\Gamma = \sum_{j=1}^{M} \gamma_j$ where the γ_j are sub-line segments (intersecting only at endpoints) of the line segments comprising Γ and such that each γ_j has length less than δ. It follows that, for $w \in K$,

$$f(w) = \sum_{j=1}^{M} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(z)}{z - w}\, dz.$$

For $1 \leqslant j \leqslant M$ and $w \notin \Gamma$, let

$$f_j(w) = \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(z)}{z - w}\, dz$$

and note that each f_j is analytic on $\mathbb{C}\backslash\{\Gamma\}$ and $\sum f_j = f$ on K. Thus, it suffices to show each f_j is uniformly approximable on K by rational functions with poles off K. Let p_j denote the midpoint of γ_j. By Lemma 18.2, each f_j is uniformly approximable by rational functions (with poles at p_j) on compact subsets of $|z - p_j| > \frac{\delta}{2}$ and hence on K. □

Remark 18.4. It is possible to improve the statement of Runge's theorem given here. For instance, the complement U of K will have at most countably many connected components. Choose any subset $E \subset U$ such that each bounded component of U contains at least one point from E. If f is analytic in a neighborhood of K, then f is uniformly approximable by rational functions with poles in A. As a special case, if U is connected (as is the case if K is simply connected), then f is uniformly approximable on K by polynomials. The remarkable theorem of Mergalyan's says: if $K \subset \mathbb{C}$ is compact and $U = \mathbb{C}\backslash K$ has finitely many components, then functions f that are continuous on K and analytic in the interior of K can be uniformly approximated on K by rational functions with poles off K. ◇

Given a compact set $K \subset \mathbb{C}$, a set $\Omega \supset K$ and a continuous function $f : \Omega \to \mathbb{C}$, let $\|f\|_K$ denote the sup norm of f (on K). The *polynomially convex hull* of K is the set,

$$\hat{K} = \{w \in \mathbb{C} : |p(w)| \leqslant \|p\|_K, \text{ for all polynomials } p\}.$$

The set K is *polynomially convex* if $K = \hat{K}$. By Remark 18.4 a compact set with connected complement (e.g. $\mathbb{D}$) is polynomially convex. The notion of the polynomially convex hull is more interesting in several complex variables.

Proposition 18.5. *Suppose K is compact and let $\mathscr{K}$ denote the complement of the unbounded component of $\mathbb{C}\backslash K$. If $K = \overline{K^\circ}$ and $\mathscr{K} = \overline{\mathscr{K}^\circ}$, then the polynomially convex hull of K is $\tilde{U} = \mathbb{C}\backslash U$, where U is the unbounded component of $\mathbb{C}\backslash K$.* †

Proof. By the maximum modulus theorem (Proposition 12.2), if p is a polynomial and C is compact with $C = \overline{C^\circ}$, then $\|p\|_C = \|p\|_{\partial C}$. Observe that $\mathscr{K} \supset K$ and $\partial\mathscr{K} \subset \partial K$. Hence, for polynomials p,

$$\|p\|_K \leqslant \|p\|_{\mathscr{K}} = \|p\|_{\partial\mathscr{K}} \leqslant \|p\|_{\partial K} = \|p\|_K.$$

It follows that $\mathscr{K} \subset \hat{K}$. On the other hand, since the complement of $\mathscr{K}$ is connected (is its unbounded component) $\mathscr{K}$ is polynomially convex; i.e., $\mathscr{K} = \hat{\mathscr{K}}$. Hence $\hat{K} \subset \mathscr{K}$. □

19. The Schwarz reflection principle

Given an open set $\Omega \subset \mathbb{C}$, let

$$\Omega^* = \{\overline{z} : z \in \Omega\}$$

and, assuming $0 \notin \Omega$, let

$$\Omega^{-*} = \{\frac{1}{\overline{z}} : z \in \Omega\}.$$

Thus Ω^* and Ω^{-*} are the reflections of Ω about the real axis and unit circle respectively.

The proof of the following lemma is left as an (easy) exercise.

Lemma 19.1. *Let $\Omega \subset \mathbb{C}$ be an open set. If $f : \Omega \to \mathbb{C}$ is analytic, then*

(i) $\check{f}(z) : \Omega^* \to \mathbb{C}$ defined by $f^*(z) = \overline{f(\overline{z})}$; and
(ii) assuming $0 \notin \Omega$, the function $\hat{f} : \Omega^{-*} \to \mathbb{C}$ defined by $\hat{f}(z) = f(\frac{1}{\overline{z}})$

are analytic. †

For notational convenience, let $\Omega_+ = \{z \in \Omega : \operatorname{Im} z \geqslant 0\}$ and $\Omega_{\leqslant 1} = \{z \in \Omega : |z| \leqslant 1\}$.

Theorem 19.2. *Let $\Omega \subset \mathbb{C}$ be an open set.*

(i) If $\Omega = \Omega^*$ and $f : \Omega_+ \to \mathbb{C}$ is continuous, the restriction of f to $\{z \in \Omega : \operatorname{Re} z > 0\}$ is analytic and if $f(z)$ is real for $z \in \Omega$ with $\operatorname{Im} z = 0$, then $g : \Omega \to \mathbb{C}$ defined by

$$g(z) = \begin{cases} \overline{f(\overline{z})}, & z \in \Omega, \quad \operatorname{Re} z < 0 \\ f(z) & z \in \Omega, \quad \operatorname{Re} z \geqslant 0 \end{cases}$$

is analytic; and

(ii) assuming $0 \notin \Omega$, if $\Omega = \Omega^{-*}$ and $f : \{z \in \Omega_{\leqslant 1} \to \mathbb{C}$ is continuous, never 0, the restriction of f to $\{z \in \Omega : |z| < 1\}$ is analytic and $|f(z)| = 1$ for $z \in \Omega$ with $|z| = 1$, then $g : \Omega \to \mathbb{C}$ defined by

$$g(z) = \begin{cases} \frac{1}{\overline{f(\frac{1}{\overline{z}})}}, & z \in \Omega, \quad |z| > 1 \\ f(z) & z \in \Omega, \quad |z| \leqslant 1 \end{cases}$$

is analytic.

Proof sketch. We prove the first statement only as the second is similar. By the pasting Lemma, g is continuous. To prove g is analytic it therefore suffices to verify the hypotheses of (Morera's) Theorem 3.1. Accordingly suppose T is an oriented triangle lying (including its *interior*) in Ω. The restriction of g to Ω intersected with either the open upper or open lower half plane is analytic by assumption and Lemma 19.1 respectively. Hence if T lies entirely in either the upper or lower half plane,

$$\int_T f = 0,$$

by Cauchy's Theorem. Since T can be written as an oriented sum of finitely many oriented triangles such that each lies in either Ω intersect the closed upper half plane or in Ω intersect the lower half plane. Without loss of generality suppose the triangle $T = [\![a, b, c]\!]$ (including its interior) lies in $\Gamma = \{z \in \Omega : \operatorname{Re} z \geqslant 0\}$ and at least one vertex lies on the real axis. Without loss of generality, assume $\operatorname{Im} c > 0$. Letting Δ denote T along with its interior, note that f is continuous, and hence uniformly continuous, on Δ. Choose sequence (a_n) and (b_n) converging to a and b respectively and so that $(T_n) = [\![a_n, b_n, c]\!] \subset \Delta \cap \{\operatorname{Im} z > 0\}$. It follows that

$$0 = \int_{T_n} f \to \int_T f.$$

□

Problem 19.1. State and prove a version of the Schwarz reflection principle where $f(z)$ is real-valued for $z \in \Omega$ with $|z| = 1$.

Problem 19.2. Prove, if $f : \mathbb{C} \to \mathbb{C}$ is entire and $|f(z)| = 1$ for $|z| = 1$, then there is an $n \in \mathbb{N}$ and unimodular constant c so that $f(z) = cz^n$.

20. Introduction to harmonic functions

Suppose $\Omega \subset \mathbb{C}$ is open. A function $u : \Omega \to \mathbb{R}$ is *harmonic* if it has continuous second partial derivatives and

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{9}$$

Proposition 20.1. *If $f : \Omega \to \mathbb{C}$ is analytic, then $u = \operatorname{Re} f$ and $v = \Im f$ are twice differentiable and satisfy the Cauchy Riemann equations. In particular both are harmonic.* †

As an example, if $f : \Omega \to \mathbb{C}$ is analytic an never 0, the $\log(|f|)$ is harmonic. To prove this statement one can write $f = u + iv$ and use $\log(|f|) = \frac{1}{2}\log(u^2 + v^2)$ to compute the second partials directly. A more abstract (and easier) argument is to observe it suffices to assume the domain of f is an open ball (and in particular simply connected) and appeal to Corollary 7.10(iv) to write $f = e^g$ for some analytic function g. Thus $|f| = \exp(\Re g)$ and $\log(|f|) = \Re g$ is harmonic.

A pair (u, v) satisfying the Cauchy-Riemann equations are *harmonic conjugates*. Similarly, if u is harmonic and (u, v) are harmonic conjugates, then v is a *harmonic conjugate* of u. It is easy to verify, if u is harmonic on a connected set, then, up to an additive constant, u has at most one harmonic conjugate.

Proposition 20.2. *Suppose G is either $\mathbb{D}$ or $\mathbb{C}$. If $u : G \to \mathbb{R}$ is harmonic, then there is an harmonic function $v : G \to \mathbb{R}$ such that $f = u + iv$ is analytic; i.e., u has a harmonic conjugate.* †

Proof. Because of the geometry of G, we may define, $v : G \to \mathbb{R}$ by

$$v(x, y) = \int_0^y \frac{\partial u}{\partial x}(x, t)\, dt - \int_0^x \frac{\partial u}{\partial y}(s, 0)\, ds.$$

First suppose $x, y \geqslant 0$ and $(x, y) \in G$. There is a $\delta > 0$ such that $R = [-\delta, x + \delta] \times [-\delta, y + \delta] \subset G$. Since the second partials of u are continuous on R, Theorem 1.3 allows for differentiating under the integral signs to obtain,

$$\begin{aligned}\frac{\partial v}{\partial x}(x, y) &= \int_0^y \frac{\partial^2 u}{\partial x^2}(x, t)\, dt - \frac{\partial u}{\partial y}(x, 0)\\ &= -\int_0^y \frac{\partial^2 u}{\partial y^2}(x, t)\, dt - \frac{\partial u}{\partial y}(x, 0)\\ &= -\frac{\partial u}{\partial y}(x, y),\end{aligned}$$

where both fundamental theorems of calculus were used. Of course,

$$\frac{\partial v}{\partial y}(x, y) = \frac{\partial u}{\partial x}(x, y).$$

The cases where not both x, y are non-negative are similar. Hence (u, v) satisfy the Cauchy-Riemann equations. It follows that $f = u + iv$ is analytic and $\operatorname{Re} f = u$. □

Lemma 20.3. *Suppose $\Omega, \Gamma \subset \mathbb{C}$ are open sets. If $h : \Gamma \to \Omega$ is analytic and $u : \Omega \to \mathbb{R}$ is harmonic, then $u \circ h : \Gamma \to \mathbb{R}$ is harmonic.* †

Proof. The lemma can be verified by direct computation. Alternately, fix a point $y \in \Gamma$ and an $r > 0$ such that $B(h(y); r) \subset \Omega$. There is a $\delta > 0$ such that $h(B(y; \delta)) \subset B(h(y); r)$. The function u has a harmonic conjugate v on $B(h(y); r)$ and thus $f = u + iv : B(y; r) \to \mathbb{C}$ is analytic. It follows that $f \circ h|_{B(y;\delta)} : B(y, \delta) \to \mathbb{C}$ is analytic. Hence, $u \circ h|_{B(y;r)} = \operatorname{Re} f \circ h|_{B(y;r)}$ is a harmonic by Proposition 20.1. Since y is arbitrary, $u \circ h$ is harmonic. □

Theorem 20.4. *If $G \subset \mathbb{C}$ is a simply connected domain and $u : G \to \mathbb{R}$ is harmonic, then there is an analytic function $f : G \to \mathbb{C}$ such that $u = \operatorname{Re} f$.*

Proof. The case $G = \mathbb{C}$ is covered by Proposition 20.2. Accordingly, suppose $G \neq \mathbb{C}$. In this case, by Theorem 16.1 (the Riemann mapping theorem) there is a one-one analytic mapping (with analytic inverse) $h : \mathbb{D} \to G$. Let $U = u \circ h$. By Lemma 20.3, U is harmonic in $\mathbb{D}$. Thus, by what has already been proved, there is a harmonic function

$V : \mathbb{D} \to \mathbb{R}$ such that $F = U + iV$ is analytic. It follows that $f = F \circ h^{-1} : G \to \mathbb{C}$ is analytic and therefore $\operatorname{Re} f = u$ is harmonic. □

20.1. **Harmonic functions on an annulus.** Fix $0 \leqslant r < 1$ and let $\mathbb{A}_r = \{r < |z| < 1\}$. Let $\mathbb{H}_r = \{\log(r) < \operatorname{Re} z < 0\}$ and note that $\exp : H_r \to \mathbb{A}_r$ is analytic and onto.

Example 20.5. The function $u : \mathbb{A}_r \to \mathbb{C}$ defined by $u(z) = \log(|z|)$ is harmonic, but not the real part of an analytic function. △

Lemma 20.6. *If $f : \mathbb{H}_r \to \mathbb{C}$ is analytic and $2\pi i$ periodic, then there is an analytic function $\tilde{f} : \mathbb{A}_r \to \mathbb{C}$ such that $\tilde{f} = f \circ \exp$.* †

Lemma 20.7. *Suppose $f : \mathbb{D} \to \mathbb{C}$ is analytic except for an isolated singularity at 0. If $\operatorname{Re} f$ is bounded above (or below), then 0 is a removable singularity.* †

Proof. Let $D = \{0 < |z| < 1\}$. Without loss of generality, we may assume $\operatorname{Re} f(z) < 0$ for all $z \in D$. Let $\mathbb{H} = \{\operatorname{Re} z < 0\}$. There is a mobius map φ mapping $\mathbb{H}$ bijectively to $\mathbb{D}$. In particular $g = \varphi \circ f : D \to \mathbb{D}$ is analytic (and bounded). Hence g extends to an analytic map $\hat{g} : \mathbb{D} \to \overline{\mathbb{D}}$. On the other hand, $\hat{g}$ is either constant or $\hat{g}(\mathbb{D})$ is open. In either case, $\hat{g}$ maps into $\mathbb{D}$. It follows that $\varphi^{-1} \circ \hat{g} : \mathbb{D} \to \mathbb{C}$ is analytic and agrees with f on D. □

Proposition 20.8. *If $h : \mathbb{A}_r \to \mathbb{R}$ is harmonic, then there is an analytic function f such that*

$$h(z) = \operatorname{Re} f(z) + c \log(|z|).$$

Moreover, in the case $r = 0$, if h is bounded, then h extends to a harmonic function on all of $\mathbb{D}$. †

Proof. Since h is harmonic and exp is analytic, by Lemma 20.3, the function $u = h \circ \exp : \mathbb{H}_r \to \mathbb{R}$ given by

$$u(z) = h(e^z)$$

is harmonic and $2\pi i$ periodic. Since $\mathbb{H}_r$ is simply connected, by Theorem 20.4, u has a harmonic conjugate v. Further, since u is $2\pi i$ periodic, so is its gradient. By the Cauchy-Riemann equations, the gradient of v is also $2\pi i$ periodic; i.e.,

$$\nabla v(z + 2\pi i) - \nabla v(z) = 0.$$

It follows that there is a constant $c \in \mathbb{R}$ such that

$$v(z + 2\pi i) - v(z) = c.$$

Define $g : \mathbb{H}_r \to \mathbb{C}$ by

$$g(z) = u(z) + iv(z) - \frac{c}{2\pi} z.$$

By construction g is analytic and 2π periodic. By Lemma 20.6, there is an analytic function $f : \mathbb{A}_r \to \mathbb{C}$ such that $g = f \circ \exp$. We conclude that $u(z) - \frac{c}{2\pi} z = \operatorname{Re} f(e^z)$; i.e.,

$$h(e^z) - \frac{c}{2\pi} \operatorname{Re} z = \operatorname{Re} f(e^z).$$

Letting $y = e^z$ gives

$$h(y) - \frac{c}{2\pi}\log(|y|) = \operatorname{Re} f(y). \tag{10}$$

The context of the moreover part of the result is the case $r = 0$ where $\mathbb{A}_r$ is the punctured disc $\{0 < |z| < 1\}$. Suppose $c \geqslant 0$ in equation (10). In this case $\operatorname{Re} f$ is bounded above and therefore 0 is a removable singularity for f. But then both h and $\operatorname{Re} f$ are bounded in $\mathbb{D}$ and therefore $c = 0$. The case $c \leqslant 0$ is similar. It follows that $h(y) = \operatorname{Re} f(y)$ extends harmonically to all of $\mathbb{D}$. ☐

21. The maximum principle

Proposition 21.1. *Suppose $\Omega \subset \mathbb{C}$ is open, $r > 0$ and $\overline{B(y;r)} \subset \Omega$. If $u : \Omega \to \mathbb{R}$ is harmonic, then*

$$u(y) = \frac{1}{2\pi}\int_0^{2\pi} u(y + re^{it})\, dt.$$

†

Proof. There is an $0 < r < R$ such that $B(y;R) \subset \Omega$. By Proposition 20.2, there is an analytic function $f : B(y;R) \to \mathbb{C}$ such that $\operatorname{Re} f = u$ (on $B(y;R)$). Thus, by Cauchy's integral formula,

$$f(y) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-y}\, dw,$$

where $\gamma(s) = a + re^{is}$, for $0 \leqslant s \leqslant 2\pi$. Hence,

$$f(y) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{is})\, ds.$$

Taking real parts completes the proof. ☐

Suppose $\Omega \subset \mathbb{C}$ is an open set and $u : \Omega \to \mathbb{R}$ is continuous. For $y \in \Omega$ and $r > 0$ such that $\overline{B(y;r)} \subset \Omega$, let

$$A_u(y;r) = A(y;r) = \frac{1}{2\pi}\int_0^{2\pi} u(y + re^{is})\, ds.$$

The function u has the *mean value property* if $u(y) = A(y;r)$ for all such y and r. Likewise u is *subharmonic* (resp. *superharmonic*) if $u(y) \leqslant A(y;r)$ (resp. $u(y) \geqslant A(y;r)$) for all such y and r.

As a provisional definition, a continuous function $u : \Omega \to \mathbb{R}$ is *locally subharmonic* if for each $y \in G$ there is an $r_y > 0$ such that $\overline{B(y;r_y)} \subset G$ and $u(y) \leqslant A_u(y;r)$ for each $0 < r < r_y$.

Theorem 21.2 (Maximum principle)**.** *If G is a domain and $u : G \to \mathbb{R}$ is (continuous and) locally subharmonic, then either u is constant, or u does not attain its supremum.*

If G is a bounded domain, $u : \overline{G} \to \mathbb{R}$ is continuous and $u|_G$ is locally subharmonic, then u attains its maximum on ∂G. In particular, if $u|_{\partial G} = 0$, then $u \leqslant 0$ on G.

Proof. Suppose u does attain its supremum in G; i.e., there is a point $y \in G$ such that $u(y) \geqslant u(z)$ for all $z \in G$. It suffices to prove $u(z) = u(y)$ for all $z \in G$. Let

$$\Omega = \{z \in G : u(z) = u(y)\}.$$

Since u is continuous, Ω is closed in G. Let $z \in \Omega$ be given. Fix $r_y > 0$ as in the hypothesis of the theorem. For $0 < r < r_y$, by subharmonicity of u,

$$\frac{1}{2\pi}\int_0^{2\pi} [u(z) - u(z + re^{is})]\, ds \leqslant 0.$$

Since the integrand is continuous and non-negative (as $z \in \Omega$), it follows that the integrand is identically zero. Thus $B(z; r_y) \subset \Omega$ and therefore, by connectedness, $\Omega = G$. □

Of course, this maximum principle for subharmonic functions is also the minimum principle for superharmonic functions. In particular, a function satisfying the mean value property satisfies both the maximum and minimum principles.

Given $\Omega \subset \mathbb{C}$, recall $\partial_\infty \Omega = \partial\Omega$ in the case Ω is bounded and $\partial_\infty\Omega = \partial\Omega \cup \{\infty\}$ in the case Ω is not bounded. Suppose $u : \Omega \to \mathbb{R}$. For a point $\infty \neq w \in \partial_\infty \Omega$,

$$\limsup_{z\to w} u(z) = \lim_{\delta\to 0+} \sup\{u(z) : 0 < |z - w| < \delta,\, z \in \Omega\}$$

and, assuming Ω is unbounded,

$$\limsup_{z\to\infty} u(z) = \lim_{C\to\infty} \sup\{u(z) : |z| > C, z \in \Omega\}.$$

As expected, $\limsup(u + v) \leqslant \limsup u + \limsup v$ and $\limsup(-u) = -\liminf u$.

Theorem 21.3. *Suppose $G \subset \mathbb{C}$ is a domain and $u, v : G \to \mathbb{R}$. If u is subharmonic, v is superharmonic and for each $w \in \partial_\infty G$,*

$$\limsup_{z\to w} u(z) \leqslant \liminf_{z\to w} v(z),$$

then $u(z) < v(z)$ for all $z \in G$ or $u = v$.

Proof. By properties of the lim sup and the hypothesis, for $w \in \partial_\infty G$,

$$\limsup_{z\to w}(u(z) - v(z)) \leqslant 0.$$

Further, $u - v$ is subharmonic. Thus, it suffices to prove if $u : G \to \mathbb{R}$ is subharmonic and

$$\limsup_{z\to w} u(z) \leqslant 0$$

for all $w \in \partial_\infty G$, then $u(z) \leqslant 0$ for all $z \in G$ and if there is a point $y \in G$ such that $u(y) = 0$, then u is identically zero. Arguing by contradiction, suppose $a \in G$ and $u(a) > 0$. Let

$$A = \{z \in G : u(z) \geqslant \frac{u(a)}{2}\}.$$

If G is not bounded, then, since $\limsup_{z\to\infty} u(z) \leqslant 0$, there is a $C > 0$ such that if $|z| > C$ and $z \in \Omega$, then

$$u(z) < \frac{u(a)}{2}.$$

Hence $A \subset \overline{B(0,C)}$. Thus, whether A is bounded or not, there is a $C > 0$ such that $A \subset \overline{B(0,C)}$. The set $B = \partial G \cap \overline{B(0,C)}$ is compact. For each $w \in B$, there is a $\delta_w > 0$ such that $u(z) < \frac{u(y)}{2}$ for $z \in G \cap B(w;\delta_w)$. The collection $\{B(w,\delta_w) : w \in B\}$ is an open cover of B. Hence, by the Lebesgue number lemma, there is a $\delta > 0$ such that for each $b \in B$ there is a w such that $B(b,\delta) \subset B(w,\delta_w)$. In particular, if $d(z,B) < \delta$, then $u(z) < \frac{u(a)}{2}$ and hence $z \notin A$. Thus,

$$A \subset \{z \in G : d(z,B) \geqslant \delta\} \cap \{z : |z| \leqslant C\} := K.$$

It follows that

$$A = \{z \in K : u(z) \geqslant \frac{u(a)}{2}\}$$

and since u is continuous and K is compact, A is compact. It follows that u attains its maximum on A; i.e., there is a point $p \in A$ such that $u(p) \geqslant u(z)$ for all $z \in A$. Thus, $u(p) \geqslant u(z)$ for all $z \in G$ and by Theorem 21.2 u is constantly equal to $u(p) > 0$ a contradiction (since then the lim sup would be constantly equal to $u(p) > 0$). Hence $u(z) \leqslant 0$ for all $z \in G$. Finally, if $u(z) = 0$ for some $z \in G$, then Theorem 21.2 implies u is identically 0. □

Example 21.4. Define $f : \mathbb{D} \to \mathbb{R}$ by

$$f(z) = (\frac{1+z}{1-z})^2$$

and let $u = \operatorname{Im} f$. Since u is the imaginary part of an analytic function, u is harmonic. Moreover, u extends to be continuous on $\mathbb{C}\backslash\{1\}$ and hence, for $|z| = 1$ and $z \neq 1$,

$$\lim_{t\to 1, 0<t<1} u(tz) = 0.$$

On the other hand, the limit above is also 0 for for $z = 1$ and $0 < t < 1$, since $f(tz)$ is real. However, $\limsup_{z\to 1} u(z) = \infty$ as can be seen by approaching 1 along the circle $\frac{1}{2} + i\frac{1}{2}e^{is}$. △

The following corollary says that a harmonic function is determined by its boundary values.

Corollary 21.5. *Suppose G is a bounded domain and $u : \overline{G} \to \mathbb{R}$ is continuous. If u satisfies the mean value property on G. If $u = 0$ on ∂G, then u is identically zero.* †

Proof. Choose $v = 0$ in Theorem 21.3 to conclude $\pm u \leqslant 0$ on G. □

Proposition 21.6. *Let $\Omega \subset \mathbb{C}$ be an open set. If $u_1, u_2 : \Omega \to \mathbb{R}$ are subharmonic, then so is $v = \max\{u_1, u_2\}$.* †

Proof. Fix $y \in \Omega$ and $r > 0$ such that $\overline{B(y;r)} \subset \Omega$ and note that for $i = 1, 2$,

$$u_i(y) \leqslant \frac{1}{2\pi} \int_0^{2\pi} u_1(y + re^{it})\, dt \leqslant \frac{1}{2\pi} \int_0^{2\pi} v(y + re^{it})\, dt.$$

Hence,

$$v(a) \leqslant \frac{1}{2\pi} \int_0^{2\pi} v(y + re^{it})\, dt.$$

□

22. The Poisson Kernel

The function $P : [0, 1) \times \mathbb{R} \to \mathbb{R}$ defined by

$$P(r,t) = P_r(t) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{int}$$

is the *Poisson kernel*.

Lemma 22.1. *For* $(r, t) \in [0, 1) \times \mathbb{R}$,

$$P_r(t) = \mathrm{Re}\left(\frac{1 + re^{it}}{1 - re^{it}}\right) = \frac{1 - r^2}{1 - 2r\cos(t) + r^2}.$$

†

Lemma 22.2 (Approximate identity)**.** *The Poisson kernel has the following properties.*

(i) $P_r(t) \geqslant 0$ *for all* $(r, t) \in [0, 1) \times \mathbb{R}$;

(ii) For $0 \leqslant r < 1$,

$$\int_0^{2\pi} P_r(t)\, dt = 1;$$

(iii) if $0 < \delta < |t| \leqslant \pi$, *then* $P_r(t) \leqslant P_r(\delta)$;

(iv) for each $\delta, \epsilon > 0$ *there exists an* $\eta > 0$ *such that if* $0 < 1 - r < \eta$ *and* $\pi \geqslant |t| \geqslant \delta$, *then*

$$\epsilon > P_r(t).$$

†

Proof. Item (i) follows from Lemma 22.1.

Item (ii) is immediate from the definition of $P_r(t)$.

Item (iii) follows from writing,

$$P_r(t) = \frac{1 - r^2}{(1 - r)^2 + 2r(1 - \cos(t))}.$$

Finally, to prove item (iv), estimate

$$0 \leqslant P_r(t) = \frac{1 - r^2}{(1 - r)^2 + 2r(1 - \cos(t))} \leqslant \frac{1 - r^2}{2r} \frac{1}{1 - \cos(\delta)}.$$

□

Proposition 22.3. *If $u : [-\pi, \pi] \to \mathbb{R}$ is continuous and periodic ($u(\pi) = u(-\pi)$), then for each $\epsilon > 0$ there exists $\delta, \eta > 0$ such that if $|\alpha - \beta| < \delta$ and $\eta > 1 - r > 0$, then*

$$\left| u(\beta) - \frac{1}{2\pi} \int_0^{2\pi} P_r(\alpha - t) u(t)\, dt \right| < \epsilon.$$

†

Proof. Extend u to all of $\mathbb{R}$ by periodicity. Let $\epsilon > 0$ be given. By uniform continuity of u, there is an $\delta > 0$ such that if $|p - q| < \delta$, then $|u(p) - u(q)| < \epsilon$. Using Lemma 22.2(iv), choose $\eta > 0$ such that $\epsilon > P_r(s)$ for $\pi \geqslant |s| \geqslant \frac{\delta}{2}$ and $\eta > 1 - r > 0$. Thus, for $\eta > 1 - r > 0$ and $|\alpha - \beta| < \frac{\delta}{2}$, and using $|(\alpha - s) - \beta| < \delta$ for $|s| < \frac{\delta}{2}$,

$$\begin{aligned}
|\int_0^{2\pi} [P_r(\alpha - t) u(t) - u(\beta)]\, dt| &= \left| \int_0^{2\pi} [P_r(s) u(\alpha - s) - u(\beta)]\, ds \right| \\
&\leqslant \int_0^{2\pi} P_r(s) \, |u(\alpha - s) - u(\beta)| \, ds \\
&= \int_{\pi \geqslant |s| \geqslant \frac{\delta}{2}} + \int_{|s| \leqslant \frac{\delta}{2}} \\
&\leqslant \epsilon \int_0^{2\pi} [|u(\alpha - s))| + |u(\beta)|]\, ds + \int_{|s| \leqslant \frac{\delta}{2}} P_r(s) |u(\alpha - s) - u(\beta)|\, ds \\
&\leqslant 2\pi (M + 1)\epsilon,
\end{aligned}$$

where M is an upper bound for $|u|$. To complete the proof, observe

$$u(\beta) - \frac{1}{2\pi} \int_0^{2\pi} P_r(\alpha - t) u(t)\, dt = \frac{1}{2\pi} \int_0^{2\pi} [P_r(t - \alpha) u(t) - u(\beta)]\, dt.$$

□

Theorem 22.4 (Dirichlet problem for $\mathbb{D}$)**.** *If $v : \partial\mathbb{D} \to \mathbb{R}$ is continuous, then there exists a continuous function $u : \overline{\mathbb{D}} \to \mathbb{R}$ such that*

(i) $u|_{\mathbb{D}}$ is harmonic; and
(ii) $u|_{\partial\mathbb{D}} = v$.

Moreover, $f : \mathbb{D} \to \mathbb{C}$ defined by

$$f(re^{i\alpha}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + re^{i(\alpha - t)}}{1 + re^{i(\alpha + t)}} v(e^{it})\, dt$$

is analytic in $\mathbb{D}$ and $u|_{\mathbb{D}} = \operatorname{Re} f$.

Proof. Define $\tilde{v} : [-\pi, \pi] \to \mathbb{R}$ by $\tilde{v}(t) = v(e^{it})$. Define $u : \overline{\mathbb{D}} \to \mathbb{R}$ by,

$$u(re^{i\alpha}) = \begin{cases} \frac{1}{2\pi} \int_0^{2\pi} P_r(\alpha - t) \tilde{v}(t)\, dt & z = re^{i\alpha},\ 0 \leqslant r < 1 \\ v(e^{i\alpha}) & z = e^{i\alpha} \end{cases}.$$

Proposition 22.3 says u is continuous at each point in $\partial\mathbb{D}$.

To complete the proof, it suffices to establish the moreover part of the theorem. Toward this end, define $g : \mathbb{D} \times \mathbb{R} \to \mathbb{C}$ by

$$g(z,t) = \frac{1+ze^{-it}}{1-ze^{-it}}\,\tilde{v}(t).$$

An application of Lemma 5.1 shows

$$f(z) = \int_0^{2\pi} g(z,t)\,dt$$

is analytic. Hence f is analytic and consequently $u|_{\mathbb{D}} = \operatorname{Re} f$ is harmonic (and in particular continuous). □

Corollary 22.5. *If $U : \overline{\mathbb{D}} \to \mathbb{R}$ is continuous and harmonic on $\mathbb{D}$, then, for $z = re^{i\alpha} \in \mathbb{D}$,*

$$U(z) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\alpha - t)U(e^{it})\,dt$$

†

Proof. Let $v = U|_{\partial\mathbb{D}}$ and let u denote the function produced by Theorem 22.4. Thus u and U are both continuous on $\partial\mathbb{D}$, are continuous on $\overline{\mathbb{D}}$ and are harmonic in $\mathbb{D}$. Therefore, by Corollary 21.5, $u = U$. □

Theorem 22.6. *If $\Omega \subset \mathbb{C}$ is open and $u : \Omega \to \mathbb{C}$ is continuous and has the mean value property, then u is harmonic.*

Proof. Fix $y \in \Omega$ and $r > 0$ such that $\overline{B(y;r)} \subset \Omega$. From Theorem 22.4, there is a continuous function w on $\overline{B(y;r)}$ such that w agrees with u on $\{|z-y| = r\}$ and w is harmonic in $B(y;r)$. In particular, w has the mean value property and therefore $u - w$ has the mean value property on $B(y;r)$, is continuous on $\overline{B(y;r)}$ and is zero on the boundary of $B(y;r)$. Thus, by Corollary 21.5, $u - w = 0$ in $B(y;r)$. Thus u is harmonic. □

22.1. **Subharmonic functions revisited.**

Proposition 22.7. *Suppose G is region and $u : G \to \mathbb{R}$ is continuous. If, u is locally subharmonic, then u is subharmonic.* †

Proof. Fix $y \in G$ and $R > 0$ such that $\overline{B(y;R)} \subset G$. We are to show $u(y) \leqslant A_u(y;R)$. Let $B = B(y;R)$. There exists a continuous function $\varphi : \overline{B} \to \mathbb{R}$ such that $\varphi|_{\partial B} = u|_{\partial B}$ and $\varphi|_B$ is harmonic. It follows that $\psi = (u-\varphi)_{\overline{B}}$ satisfies the hypothesis of Theorem 21.2. Thus, $u \leqslant \varphi$ on B since $\psi|_{\partial B} \leqslant 0$. Consequently,

$$u(y) \leqslant \varphi(y) = A_\varphi(y;R) = A_u(y;R).$$

□

Proposition 22.8. *Suppose $G \subset \mathbb{C}$ is a bounded domain, $u : G \to \mathbb{R}$ is subharmonic and $B = \overline{B(y;r)} \subset G$. Let $\hat{u}$ denote the solution to the Dirichlet problem*

(a) $\hat{u}: B \to \mathbb{R}$ *is continuous;*
(b) $\hat{u}|_{\partial B} = u|_{\partial B}$*; and*
(c) $\hat{u}|_{B(y;r)}$ *is harmonic.*

The function $v: G \to \mathbb{R}$ *defined by*

$$v(z) = \begin{cases} u(z) & z \in G \backslash B(y;r) \\ \hat{u}(z) & z \in B \end{cases}$$

is subharmonic and $u(y) \leqslant v(y)$.

In particular, if $u: \overline{G} \to \mathbb{R}$ *is continuous,* $u|_G$ *is subharmonic and* $\overline{B(y;r)} \subset G$*, then there is a continuous function* $v: \overline{G} \to \mathbb{R}$ *such that* $v|_{B(y;r)}$ *is harmonic,* $u(y) \leqslant v(y)$ *and* $v|_{\partial G} = u|_{\partial G}$. †

Proof. The strategy is to show v is locally subharmonic and apply Proposition 22.7. If $w \in B(y;r)$, then $v(s) = A_v(w;s)$ for all $0 < s < r - |y-w|$ since v is harmonic on $B(w; r - |y-w|)$. Likewise, if $w \in G \backslash \overline{B(y;r)}$, then

$$A_v(w;s) = A_u(w;s) \geqslant u(w) = v(w),$$

for $0 < s < r_w := \min\{|y-w| - r, \operatorname{dist}(w, \partial G)\}$, since $u = v$ on $B(w; r_w)$.

Now suppose $w \in \partial B(y;r)$ and observe $v(z) \geqslant u(z)$ in $B(y;r)$ by Theorem 21.2. Thus $u \leqslant v$. For s sufficiently small,

$$A_v(w;s) \geqslant A_u(w;s) \geqslant u(w) = v(w).$$

□

23. Harnack's inequality

Remark 23.1. Let $\mathrm{Har}_0^+(\mathbb{D})$ denote the set of harmonic functions u on $\mathbb{D}$ with positive real part ($\operatorname{Re} u(z) \geqslant 0$ for all $z \in \mathbb{D}$) and normalized by $u(0) = 1$ viewed as a subset of $\mathrm{Har}(\mathbb{D})$, the set of harmonic functions on $\mathbb{D}$. It is immediate that $\mathrm{Har}_0^+(\mathbb{D})$ is a convex set. It is a bit of an exercise to show, for each $\alpha \in \mathbb{R}$, the function

$$u_\alpha(z = re^{i\alpha}) = \operatorname{Re} \frac{1 + e^{i(\alpha - t)}}{1 - e^{i(\alpha - t)}} = \operatorname{Re} \frac{1 + ze^{-it}}{1 - ze^{-it}}$$

is an extreme point of the $\mathrm{Har}_0^+(\mathbb{D})$. Theorem 22.4 says, if $u \in \mathrm{Har}_0^+(\mathbb{D})$ extends to be continuous on $\overline{\mathbb{D}}$, then u is not an extreme point. As a generalization of Theorem 22.4, if $u \in \mathrm{Har}_0^+(\mathbb{D})$, then there is a probability measure μ on $\partial\mathbb{D}$ such that, for $z \in \mathbb{D}$,

$$u(re^{i\alpha}) = \frac{1}{2\pi} \int P_r(\alpha - t)\, d\mu(t).$$

In particular, the functions u_α are exactly the extreme points of $\mathrm{Har}_0^+(\mathbb{D})$ (and they correspond to μ equal to point mass at $e^{i\alpha}$). ◇

Theorem 23.2 (Harnack's inequality)**.** *Fix $y \in \mathbb{C}$ and $r > 0$. If $u : \overline{B(y;r)} \to \mathbb{R}$ is continuous, harmonic on $B(y;r)$ and $u \geqslant 0$ (pointwise), then for each $0 \leqslant r < R$ and $\alpha \in \mathbb{R}$,*

$$\frac{R-r}{R+r}u(a) \leqslant u(a+re^{i\alpha}) \leqslant \frac{R+r}{R-r}u(a).$$

Proof. Without loss of generality suppose $a = 0$ and $R = 1$. In this case, by Theorem 22.4,

$$u(re^{i\alpha}) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\alpha - t)u(e^{it})\,dt.$$

Using,

$$P_r(\alpha - t) = \frac{1-r^2}{1-2r\cos(\alpha - t) + r^2} = \frac{1-r^2}{|e^{it} - re^{i\alpha}|^2}$$

estimate

$$\frac{1-r}{1+r} \leqslant P_r(\alpha - t) \leqslant \frac{1+r}{1-r}.$$

Substituting these inequalities into the integral representation for $u(re^{i\alpha})$ and using the mean value property of u completes the proof. □

Given an open set $\Omega \subset \mathbb{C}$, let $\mathrm{Har}(\Omega)$ denote the set of harmonic function on Ω viewed a subspace of the metric space $C(\Omega, \mathbb{C})$ of continuous functions on Ω (with the topology of uniform convergence on compact sets).

Proposition 23.3. *The (metric) subspace* $\mathrm{Har}(\Omega)$ *of* $C(\Omega, \mathbb{C})$ *is complete.* †

Proof. Since $C(\Omega, \mathbb{C})$ is complete, it suffices to show $\mathrm{Har}(\Omega)$ is closed. To this end, suppose (u_n) is a sequence from $\mathrm{Har}(\Omega)$ that converges to some $u \in C(\Omega, \mathbb{C})$. Fix an open set $U \subset \Omega$ such that $K = \overline{U} \subset \Omega$. In particular, (u_n) converges to u uniformly on K from which it immediately follows that u has the mean value property on U. Since u is also continuous, by Theorem 22.6, u is harmonic on U. Hence u is harmonic. □

Theorem 23.4 (Harnack)**.** *Suppose G is a domain and (u_n) is a sequence from* $\mathrm{Har}(G)$*. If (u_n) is pointwise increasing, then either (u_n) converges uniformly on compact sets to ∞ or (u_n) converges in* $\mathrm{Har}(G)$*.*

Proof. For each $z \in G$, then sequence $(u_n(z))$ converges to some $u(z)$, either ∞ or a real number. Let

$$F = \{z \in G : u(z) \in \mathbb{R}\},\ I = \{z \in G : u(z) = \infty\}.$$

Fix $y \in G$ and an $R > 0$ such that $\overline{B(y;R)} \subset G$. For $0 < r < R$ and all n Theorem 23.2 gives

$$\frac{R-r}{R+r}u_n(y) \leqslant u_n(y+re^{i\alpha}) \leqslant \frac{R+r}{R-r}u_n(y). \tag{11}$$

It follows that if $y \in F$, then $B(y;R) \subset F$; and if $y \in I$, then $B(y;R) \subset I$. Thus both F and I are open. By connectedness of G, one of these sets is empty and the other is G.

Now suppose $G = F$. From

$$\frac{R-r}{R+r}u_n(y) - u(y) \leqslant u_n(y + re^{i\alpha}) - u(y) \leqslant \frac{R+r}{R-r}u_n(y) - u(y),$$

it follows that u is continuous at y. By the monotone convergence theorem and the fact that each u_n has the mean value property, for each $y \in G$ and $r > 0$ such that $\overline{B(y;r)} \subset G$,

$$u(y) = \lim_{n\to\infty} u_n(y) = \lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} u_n(y + re^{it})\,dt = \frac{1}{2\pi}\int_0^{2\pi} u(y + re^{it})\,dt.$$

Thus u is continuous and has the mean value property. By Theorem 22.6, u is harmonic.

Finally, to prove that the convergence is uniform on compact sets, observe that $v_n = u - u_n \geqslant 0$ and harmonic and converges to 0 pointwise. Thus using Theorem 23.2, equation (11) applied to v_n shows that for each point $y \in G$ there is an $s > 0$ such that v_n converges to 0 uniformly on $\overline{B(y;s)}$. Thus v_n converges to 0 uniformly on each compact subset of G. □

24. The Dirichlet Problem

A bounded domain $G \subset \mathbb{C}$ is a *Dirichlet domain* if foreach continuous function $f : \partial G \to \mathbb{R}$ there is a continuous function $u : \overline{G} \to \mathbb{R}$ such that $u|_G$ is harmonic and $u|_{\partial G} = f$. (It is possible to work with unbounded domains G.) Theorem 22.4 says that $\mathbb{D}$ is a Dirichlet domain.

Proposition 24.1. *The punctured disk* $D = \{0 < |z| < 1\}$ *is not a Dirichlet domain.* †

Proof. Note that $\partial D = \{0\} \cup \{|z| = 1\}$ and $\overline{D} = \overline{\mathbb{D}}$. Suppose $u : \overline{D} \to \mathbb{R}$ is a continuous function such that u is harmonic on D and $u(z) = 0$ for $|z| = 1$. In particular, $u|_D$ is a bounded harmonic function on D. By Proposition 20.8,$u|_D$ extends to a harmonic function on $\mathbb{D}$ and thus, by continuity, $u|_{\mathbb{D}}$ is harmonic. Thus u is continuous on $\mathbb{D}$, $u|_{\partial\mathbb{D}} = 0$ and $u|_{\mathbb{D}}$ is harmonic. Thus, by Corollary 21.5, $\tilde{u}$ is identically 0 and in particular, $u(0) = 0$. It follows that there does not exist a solution to the Dirichlet problem on D with continuous boundary data: find $u : \overline{D} \to \mathbb{R}$ such that $u(z) = 0$ for$|z| = 1$ and $u(0) = 1$. □

24.1. **The method of Perron.** Let $\mathscr{S} = \mathscr{S}(G) \subset C(\overline{G})$ denote the set of continuous functions $u : \overline{G} \to \mathbb{R}$ such that $u|_G$ is subharmonic. Fix, for the remainder of this section, a continuous $f : \partial G \to R$. The lower *Perron family* for f is the set

$$\mathscr{S}_f(G) = \{u \in \mathscr{S}(G) : u|_{\partial G} \leqslant f\}.$$

By the maximum principle, Theorem 21.2, for $z \in G$,

$$\mathfrak{p}_f(z) = \sup\{u(z) : u \in \mathscr{S}_f(G)\} \leqslant M,$$

where M is the maximum of $f(z)$ on ∂G. The function$\mathfrak{p}_f : G \to \mathbb{R}$ is the *Perron solution* for f or the Perron lower solution for f.

Theorem 24.2. *The function $\mathfrak{p}_f$ is harmonic.*

Proof. Fix $y \in G$ and choose $r > 0$ such that $\overline{B(y;r)} \subset G$. There is a sequence (u_n) from$\mathscr{S}_f(G)$ such that $(u_n(y))$ converges to $\mathfrak{p}_f(y)$.

By replacing u_n with $\max\{u_1, \ldots, u_n\}$ (pointwise) we may assume, in view of Proposition 21.6, (u_n) isan increasing sequence. By Proposition 22.8 it may also be assumed that each u_n is harmonic in $B(y;r)$. Let u denote the pointwise limit of the sequence (u_n). In particular, $u \leqslant \mathfrak{p}_f$ in G and $u|_{\partial G} \leqslant f$. ByTheorem 23.2 (Harnack), $(u_n|_{B(y;r)})$ converges uniformly on compacts sets to $u|_{B(y;r)}$ and $u|_{B(y;r)}$ is harmonic and $u(y) = \mathfrak{p}_f(y)$.

Now fix $p \in B(y;r)$. And again choose a sequence $(\widetilde{u}_n)_n$ from $\mathscr{S}_f(G)$ such that $(\widetilde{u}_n(p))$ converges to $\mathfrak{p}_f(p)$. By replacing $\widetilde{u}_n$ by $\max\{\widetilde{u}_n, u_n\}$ we mayassume $u_n \leqslant \widetilde{u}_n$ (pointwise). As before, we can assume $(\widetilde{u}_n)$ is pointwise increasing and each $\widetilde{u}_n$ is harmonic on $B(y;r)$. Let $\widetilde{u}$ denote the pointwise limit of $(\widetilde{u}_n)$.As before $\widetilde{u} \leqslant \mathfrak{p}_f$ in G and $\widetilde{u}|_{B(y;r)}$ is harmonic. Summarizing, converges uniformly on compact sets to a harmonic function $\widetilde{u} : B(y,r) \to \mathbb{R}$. Thus,

(a) $u \leqslant \widetilde{u}$ on G;
(b) $\widetilde{u} \leqslant \mathfrak{p}_f$ on G;
(c) $\widetilde{u}(y) = u(y)$;
(d) $\widetilde{u}(p) = \mathfrak{p}_f(p)$;
(e) u and $\widetilde{u}$ are harmonic on $B(y;r)$.

Since $\widetilde{u} - u \geqslant 0$ is harmonic on $B(y;r)$ and is 0 at y, it follows that $\widetilde{u} = u$ on $B(y;r)$ and therefore $u(p) = \widetilde{u}(p) = \mathfrak{p}_f(p)$. Since $p \in B(y;r)$ was arbitrary, $\mathfrak{p}_f = u$ on $B(y,r)$. Thus $\mathfrak{p}_f$ is harmonic on $B(y;r)$. Since y was arbitrary, $\mathfrak{p}_f$ is harmonic on all of G. □

24.2. **Geometric sufficient conditions.** A *barrier at a point* $b \in \partial G$ is a continuous function $\varphi : \overline{G} \to \mathbb{C}$ such that $\varphi|_G$ is harmonic, $\varphi(b) = 0$ and $\varphi(z) < 0$ for $z \in \overline{G}\backslash\{b\}$. If G is a Dirichlet domain, then$G$ has a barrier at each point of ∂G (by Uryshon's Lemma for perfectly normal (e.g. metric) spaces).

Theorem 24.3. *If G is a bounded domain with a barrier at $b \in \partial G$, then*

$$\lim_{z \to b} \mathfrak{p}_f(z) = f(b).$$

Proof. Without loss of generality assume $f(b) = 0$. Let $M = \max\{|f(z)| : z \in \partial G\}$.

Let φ be a barrier at b. Let $\epsilon > 0$ be given. There is an open set $b \in U \subset \partial G$ such that $|f(z) - f(b)| < \epsilon$ for $z \in U$. On the set $\partial G\backslash U$ the function φ takes negative values and achieves its maximum which, by scaling, we can assume is -1. Consider the harmonic function $\psi : \overline{G} \to \mathbb{R}$ definedby

$$\psi(z) = \epsilon - M\varphi(z).$$

Note that $\psi(z) > \epsilon > f(z)$ on the set U. On the otherhand, on $\partial G\backslash U$ where $-\varphi(z) \geqslant 1$, we also have $\psi(z) \geqslant M \geqslant f(z)$. Thus, $\psi(z) \geqslant f(z)$ for $z \in \partial G$. Hence $\psi(z) \geqslant u(z)$ for all $z \in \overline{G}$ and $u \in \mathscr{S}_f(G)$ by the maximum principle. Thus $\psi(z) \geqslant \mathfrak{p}_f(z)$ for all $z \in G$ and consequently,

$$\limsup_{z \to b} \mathfrak{p}_f(z) \leqslant \lim_{z \to b} \psi(z) = \epsilon.$$

To prove the reverse inequality, consider the harmonic function $\psi : \overline{G} \to \mathbb{R}$ by

$$\psi(z) = M\varphi(z) - \epsilon.$$

Evidently $\psi(z) < -\epsilon$ for $z \in U$ and hence $\psi(z) < f(z)$ for $z \in U$. On the other hand, if $z \in \partial G \backslash U$, then $M\varphi(z) \leqslant -M$ and hence $\psi(z) \leqslant f(z)$. It follows that $\psi \in \mathscr{S}_f(G)$ and hence$\psi(z) \leqslant \mathfrak{p}_f(z)$ for $z \in G$. Therefore,

$$-\epsilon = \lim_{z \to b} \psi(z) \leqslant \liminf_{z \to b} \mathfrak{p}_f(z).$$

We conclude that $\lim_{z \to b} \mathfrak{p}_f(z)$ exists and is equal to$0 = f(b)$. □

Corollary 24.4. *The bounded domain $G \subset \mathbb{C}$ is a Dirichlet domain if and only if G has a barrier at each point of ∂G.* †

The following result gives an easily applied and fairly general sufficient condition for G to be a Dirichlet domain.

Proposition 24.5. *Let $G \subset \mathbb{C}$ be a domain and suppose $b \in \partial G$. If there is a point a such that $\rrbracket b, a \rrbracket \subset \mathbb{C} \backslash \overline{G}$, then there is a barrier at b.* †

Proof. Let $D = \mathbb{C} \backslash \llbracket b, a \rrbracket$ and let $D_+ = D \cup \{b\} = \mathbb{C} \backslash \rrbracket b, a \rrbracket$. In particular, $\overline{G} \subset D_+$ and $G \subset D$. There is a continuous map $\psi : D_+ \to \mathbb{C}$ such that $\psi|_D$ isanalytic, $\psi(b) = 0$ and $\psi(D) = \mathbb{H} = \{z : \operatorname{Im} z < 0\}$. (Follow a Mobius map taking a to 0 and b to $-\infty$ and the segment $\llbracket a, b \rrbracket$ to the negative real axis with the function $-\sqrt{z}$ using the principle branch of the log.) The function $\varphi = \operatorname{Im} \psi|_{\overline{G}}$ is a barrier at$b$. □

It seems the best one can do with the Perron approach is the following result, which we will not prove.

Theorem 24.6. *Suppose $G \subset \mathbb{C}$ is a domain (not necessarily bounded). If each component of the complement of G contains at least two points, then G is a Dirichlet domain.*

Corollary 24.7. *If $G \subset \mathbb{C}$ is simply connected, then G is a Dirichlet domain.* †

Remark 24.8. The situation in higher dimensions ($n \geqslant 3$) is far more complicated. ⋄

25. Green's function

Let G denote a bounded domain and suppose $y \in G$. A function $g_y : \overline{G} \backslash \{y\} \to \mathbb{R}$ such that

(a) g_y restricted to $G \backslash \{y\}$ is harmonic;
(b) $g_y = 0$ on ∂G; and
(c) there is a harmonic function f on $G \backslash \{y\}$ such that $f(z) = g_y(z) + \log(|z - y|)$ on $G \backslash \{y\}$

is called a *Green's function* for y.

Proposition 25.1. *Let G be a bounded domain and suppose $y \in G$ and suppose G has greens function g_y for y.*

(i) g_y is unique (and hence is the green's function); and
(ii) $g_y > 0$ on $G \backslash \{y\}$.

†

Proposition 25.2. *If G is a bounded Dirichlet domain, then there is a Green's function for each $y \in G$.* †

Proof. Fix $y \in G$. Let u denote the solution of the Dirichlet problem $u(z) = \log(|z-y|)$ on ∂G and define $g : \overline{G}\backslash\{y\} \to \mathbb{R}$ by $g(z) = u(z) - \log(|z-y|)$. □

Suppose $\partial G = \{\gamma\}$ is the trace of a continuously differentiable simple closed curve γ and G is a Dirichlet region. Let $g(z, y)$ denote the Green's function $g_y(z)$. If $f : \partial G \to \mathbb{R}$ is continuous, then the solution to the Dirichlet problem with boundary data f is

$$u(y) = \int_\gamma f(z) \frac{\partial g}{\partial n}(z, y)\, ds$$

where the derivative is the normal derivative and s is arclength. A proof of this statement is left to the interested reader. (Shockingly it involves Green's Theorem.)

In the special case $G = \mathbb{D}$, one easily checks that the Green's function is $g(z, y) = \log(|\varphi_y(z)|)$ where $\varphi_y(z) = \frac{z-y}{1-\overline{y}z}$. Write $y = re^{it}$ and $z = ue^{i\theta}$. In this case, the normal derivative of $g(z, y)$ on the boundary is the derivative with respect to u evaluated at $u = 1$. Writing $g(z, y)$ as $\frac{1}{2}(\log(|z-y|^2) - \log(|1-\overline{y}z|^2))$ we see this normal derivative is

$$\begin{aligned}
&\frac{1}{2}\Big[\frac{(2u - e^{i\theta}\overline{y} - e^{-i\theta}y) - (2u|y|^2 - -e^{i\theta}\overline{y} - e^{-i\theta}y)}{|1-\overline{y}ue^{i\theta}|^2}\Big]|_{u=1} \\
=&\frac{1-|y|^2}{|1-\overline{y}e^{i\theta}|^2} \\
=&P_r(t-\theta),
\end{aligned}$$

where $P_r(t)$ is the Poisson kernel. In this case $ds = \frac{1}{2\pi}d\theta$. Compare with Theorem 22.4.

26. Jensen's Formula

Theorem 26.1 (The Poisson-Jensen formula). *Suppose $\Omega \subset \mathbb{C}$ is open, $f : \Omega \to \mathbb{C}$ is analytic, $r > 0$ and $\overline{B(0;r)} \subset \Omega$. Let $a_1, \dots, a_k$ denote the zeros of f in $B(0;r)$ counted according to multiplicity. If f $|z| < r$ and $f(z) \neq 0$, then,*

$$\log(|f(z)|) = -\sum_{\ell=1}^{k} \log\left(\frac{r^2 - \overline{a_\ell}z}{r(z-a_\ell)}\right) + \frac{1}{2\pi}\int_0^{2\pi} \Re(\frac{re^{it}+z}{re^{it}-z}) \log(|f(re^{it})|\, dt.$$

The result of the following problem will be needed in the sequel. It gives an initial indication between the location of the zeros of an entire function and its rate of growth.

Problem 26.1. If f is entire and $f(0) = 1$, then $log(2)\, n(r) \leqslant M(2r)$.

We will prove Theorem 26.1 assuming $r = 1$ in which case $B(0;r) = \mathbb{D}$. For notational convenience, let

$$K(z,t) = \Re \frac{1 + ze^{-it}}{1 - ze^{-it}}.$$

Thus, writing $z = se^{i\alpha}$, K and the Poisson kernel are related by $K(z,t) = P_s(\alpha - t)$.

Lemma 26.2. *For $0 < s < 1$ and $|t| \leqslant \frac{\pi}{6}$,*

$$1 \geqslant |1 - se^{it}|^2 \geqslant |s|\, |1 - e^{it}|^2$$

†

Proof. Compute,

$$\begin{aligned} 1 - |1 - se^{it}|^2 =& 1 - \big(1 - 2s\cos(t) + s^2\big) \\ =& s(2\cos(t) - s) \geqslant 0 \end{aligned}$$

for $\cos(t) \geqslant \frac{1}{2}$. Likewise,

$$|1 - se^{it}|^2 - s\,|1 - e^{it}|^2 = (1 - s)^2$$

for all t. □

Lemma 26.3. *The function $\lambda : [-\frac{\pi}{6}, -\frac{\pi}{6}] \to \mathbb{R}$ defined by $\lambda(t) = \big|\log(|1 - e^{it}|)\big|$ is L^1.* †

Proof. Observe $|1 - e^{it}|^2 = 2(1 - \cos(t))$. By considering the power series expansion for $\cos(t)$, there exists $\eta > 0$ and $1 \geqslant \delta > 0$ and an entire function f such that $|f(z)| \geqslant \eta$ for $|z| \leqslant \delta$ and $1 - \cos(t) = t^2 f(t)$ for $t \in \mathbb{R}$. Thus the result follows from

$$\int_0^1 |\log(t)|\, dt = -\int_0^1 -\log(t)\, dt = [t(\log(t) - 1)]_0^1 = 1.$$

□

Lemma 26.4. *For $|z| < 1$,*

$$\lim_{s \to 1, 0 < s < 1} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} K(z,t) \log(|1 - se^{it}|)\, dt = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} K(z,t) \log(|1 - e^{it}|)\, dt.$$

†

Proof. For $|t| \leqslant \frac{\pi}{6}$ and $0 < s < 1$ and $|z| < 1$, by Lemma 26.2,

$$-K(z,t) \log(|1 - e^{it}|) \geqslant -K(z,t) \log(|1 - se^{it}|) \geqslant 0.$$

Using Lemma 26.3, we can apply dominated convergence to complete the proof. □

Lemma 26.5. *For $|z| < 1$,*

$$\log(|1-z|) = \frac{1}{2\pi}\int_0^{2\pi} K(z,t)\,\log(|1-e^{it}|)\,dt.$$

†

Proof. From Lemma 26.4 it follows that

$$\lim_{s\to 1, 0<s<1}\int_{-\pi}^{\pi} K(z,t)\log(|1-se^{it}|)\,dt = \int_{-\pi}^{\pi} K(z,t)\log(|1-e^{it}|)\,dt.$$

On the other hand, $\log(|1-sz|)$ is continuous on $\overline{\mathbb{D}}$ and harmonic in $\mathbb{D}$ and therefore, by Theorem 22.4,

$$\log(|1-sz|) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K(z,t)\log(|1-se^{it}|)\,dt.$$

□

Proof of Poisson-Jensen. If f has no zeros in a neighborhood of $\overline{\mathbb{D}}$,then

$$\log(|f(z)|) = \frac{1}{2\pi}\int K(z,t)\,\log(|f(e^{it})|)\,dt. \tag{12}$$

follows immediately from Theorem 22.4.

Now suppose $y \in \partial\mathbb{D}$ and f has no zeros in aneighborhood of $\overline{\mathbb{D}}$ except at y. Let $g = \frac{f}{z-y}$. Hence g now has no zeros in a neighborhood of $\overline{\mathbb{D}}$ and therefore by Lemma 26.5,

$$\begin{aligned}\log(|f(z)|) =& \log(|g(z)|) - \log(|y-z|)\\ =& \frac{1}{2\pi}\left[\int K(z,t)[\log(|g(e^{it})|) - \log(|y-e^{it}|)]\,dt\right]\\ =& \frac{1}{2\pi}\int K(z,t)\,\log(|f(e^{it})|)\,dt.\end{aligned}$$

Inducting, it now follows that if f has no zeros in $\mathbb{D}$, then the equality (12) holds.

Now suppose $f(z) \neq 0$ and f has zeros $a_1,\dots,a_k$ in $\mathbb{D}$ counted with multiplicity. Let B denote the Blaschke factor

$$B(z) = \prod_{\ell=1}^{k}\frac{z-a_\ell}{1-\overline{a_\ell}z}$$

and let $F(z) = f(z)B(z)^{-1}$. Hence F has no zeros in $\mathbb{D}$ and thus equation (12) holds with F in place of f. Using $|F| = |f|$ for $|z| = 1$ and

$$\log(|F(z)|) = \log(|f(z)|) - \sum_{\ell=1}^{k}\log(\frac{|z-a_\ell|}{|1-\overline{a_\ell}z|}).$$

Combining this last identity with the equality of equation (12) completes the proof. □

For an entire function f and $r > 0$, let $M(r) = M_f(r) = \max\{|f(z)| : |z| = r\}$ and let $n(r) = n_f(r)$ denote the number of zeros of f in $B(0;r)$.

27. Entire functions of finite genus

Recall the Weierstrass factorization Theorem. Let f be an entire functions with non-zero zeros (a_j) counted with multiplicity. The factorization of Theorem 17.12 requires a choice of nonnegative integers p_n such that

$$\sum_j \left(\frac{r}{|a_j|}\right)^{p_n+1}$$

converges.

To see that there is a relation with the previous section, observe that Jensen's formula (the case of $z = 0$ in the Poisson-Jensen formula) bounds the growth of the zeros of f in $B(0;r)$ in terms of the modulus of f on $\{|z| = r\}$.

The entire function f has *finite rank* if there is a non-negative integer p such that

$$\sum_{h=1}^{\infty} |a_j|^{-(p+1)} \tag{13}$$

converges. If f has only finitely many zeros, its rank is -1. Otherwise, the *rank* of f is the smallest p such that the series converges. If f has rank p, then one can choose $p_n = p$ in Theorem 17.12 to obtain the *standard form* or *standard factorization*,

$$f(z) = z^m e^{g(z)} P(z), \tag{14}$$

where g is entire, m is a non-negative integer (the order of the zero of f at 0) and

$$P(z) = \prod_{j=1}^{\infty} E_p\left(\frac{z}{a_j}\right).$$

Moreover, g is now uniquely determined up to an additive multiple of $2\pi i$. Hence, we may make the following definition. If the g (in the standard factorization of f in equation (14)) is a polynomial, then g has *finite genus* and the *genus* of f is the maximum of the degree of g and the rank p of f.

Proposition 27.1. *If f is an entire function of finite genus μ, then for each $\alpha > 0$ there is an $r > 0$ such that, for all $|z| > r$,*

$$|f(z)| \leqslant \exp(\alpha |z|^{\mu+1}).$$

†

We break the proof down into several lemmas.

Lemma 27.2. *Suppose ν is a positive integer. For each $A, \epsilon > 0$ there is an $R > 0$ such that $|E_\nu(z)| \leqslant A|z|^{\nu+\epsilon}$ for all $|z| > R$.* †

Proof. Estimate,

$$\log(|E_\nu(z)|) \leqslant \log(|1-z|) + \sum_{j=1}^{\nu} \frac{|z|^j}{j}.$$

Given $A > 0$ there is an $R > 0$ such that $\log(|1-z|) + \sum_{j=1}^{\nu} \frac{|z|^j}{j} \leqslant A|z|^{\nu+\epsilon}$. □

Lemma 27.3. *If ν is a positive integer and $\epsilon > 0$, then there exist positive numbers B and M such that M such that, for all z,*

$$\log(|E_\nu(z)|) \leqslant M\,|z|^{\nu+1}$$

and

$$\log(E_\nu(z)) \leqslant B|z|^{\nu+\epsilon}, \quad |z| \geqslant \frac{1}{2}$$

†

Proof. Recall the power series expansion

$$-\log(1-z) = \sum_j \frac{z^j}{j}, \quad |z| < 1.$$

Thus, for $|z| \leqslant \frac{1}{2}$,

$$\begin{aligned} \log(|E_\nu(z)|) &= \operatorname{Re}\big(\log(1-z) + \sum_{j=1}^{\nu} \frac{z^j}{j}\big) \\ &= \operatorname{Re}(-\sum_{j=\nu+1}^{\infty} \frac{z^j}{j}) \\ &\leqslant \sum_{j=\nu+1}^{\infty} \frac{|z|^j}{j} \\ &\leqslant |z|^{\nu+1} \frac{2}{\nu+1}. \end{aligned}$$

Applying Lemma 27.2 with $A = 2$ produces an $R > 0$ such that $\log(|E_\nu(z)|) \leqslant A|z|^{\nu+1}$ for $|z| > R$.

On the set $\mathbb{A} = \{\frac{1}{2} \leqslant |z| \leqslant R\}$ the function $\log(|E_\nu(z)|)$ is continuous except for a singularity at 1 where it diverges to $-\infty$. Hence $\log(|E_\nu(z)|)$ is bounded by some multiple of $|z|^{\nu+1}$ on $\mathbb{A}$ and the proof is complete. □

Lemma 27.4. *Suppose (a_j) is a sequence of non-zero complex numbers, ν is a nonnegative integer and*

$$\sum_{j=1}^{\infty} |a_j|^{-(\nu+1)}$$

converges. Let

$$Q(z) = \prod_{j=1}^{\infty} E_\nu(\frac{z}{a_j}).$$

For each $\alpha > 0$ there is an $R > 0$ such that $\log(|Q(z)|) \leqslant \alpha\,|z|^{\nu+1}$ for $|z| > R$. †

Proof. Let $\alpha > 0$ be given. Choose M as in Lemma 27.3 based upon ν. There is an N so that

$$\sum_{j=N+1}^{\infty} |a_j|^{-(\nu+1)} \leqslant \frac{\alpha}{2M}.$$

In particular,

$$\sum_{j=N+1}^{\infty} \log(|E_\nu(\frac{z}{a_j})|) \leqslant \frac{\alpha}{2}|z|^{\nu+1}. \tag{15}$$

Choose $A = \frac{\alpha}{2N}$ in Lemma 27.2 to obtain an $R_1 > 0$ such that, for $|z| > R_1$,

$$\log(|E_\nu(z)|) \leqslant \frac{\alpha}{2N}|z|^{\nu+1}.$$

Let $R_2 = \max\{|a_j| : 1 \leqslant j \leqslant N\}\, R_1$. Thus,

$$\sum_{j=1}^{N} \log(|E_\nu(\frac{z}{a_j})|) \leqslant N\frac{\alpha}{2N}\,|z|^{\nu+1} = \frac{\alpha}{2}|z|^{\nu+1}. \tag{16}$$

Combining the inequalities of equations (15) and (16) gives

$$\log(|Q(z)|) \leqslant \alpha\,|z|^{\nu+1}.$$

□

Proof of Proposition 27.1. Express f as in equation (14). Let ν denote the rank of P. In particular, $\nu \leqslant \mu$. By Lemma 27.4, there is an $R_1 > 1$ such that $|P(z)| \leqslant \alpha|z|^{\mu+1} \leqslant \alpha|z|^{\nu+1}$ for $|z| > R_1$.

Writing $g = \sum_{j=0}^{\mu} g_j z^j$,

$$\log(|z^m \exp(h(z))|) \leqslant m\log(|z|) + \sum |g_j|\,|z|^j$$

there is an $R_2 > 0$ such that

$$\log(|z^m \exp(h(z))|) \leqslant \frac{\alpha}{2}\,|z|^{\mu+1}, \quad |z| > R_2.$$

Choose $R = \max\{R_1, R_2\}$ and take exponentials to complete the proof. □

Problem 27.1. Suppose (a_j) is a sequence of non-zero numbers, $0 \geqslant \rho < p+1$ and

$$A = \sum_{j=1}^{\infty} |a_j|^\rho < \infty.$$

Suppose $0 < |a_1| \leqslant |a_2| \leqslant \dots$. Fix $z \in \mathbb{C}$ and choose N so that $|\frac{z}{a_j}| \leqslant \frac{1}{2}$ for $j > N$ and $|\frac{z}{a_j}| > \frac{1}{2}$ for $j \leqslant N$. Show

$$\sum_{j=N+1}^{\infty} \log(|E_p(\frac{z}{a_j})|) \leqslant A|z|^\rho.$$

Now show there exists a C such that for all z,

$$\sum_{j=1}^{\infty} \log(|E_p(\frac{z}{a_j})|) \leqslant C|z|^{\rho}.$$

28. Entire functions of finite order

An entire function f has *finite order* if there exists an $a > 0$ and $R \geqslant 1$ such that

$$|f(z)| \leqslant \exp(|z|^a)$$

for $|z| > R$. It is evident that if $b \geqslant a$, then $|f(z)| \leqslant \exp(|z|^b)$ for $|z| > R$ too. If f has finite order, then

$$\lambda = \inf\{a > 0 : \exists R \geqslant 1 \text{ such that} |f(z)| \leqslant \exp(|z|^a) \text{ for all } |z| > R\}$$

is the *order of the entire function* f. In particular, if λ is the order of f and $b > \lambda$, then there is an $R > 0$ such that $|f(z)| \leqslant \exp(|z|^b)$ for $|z| > R$ and if $0 < c < \lambda$, then for each $R \geqslant 1$ there is a z such that $|z| > R$ and $|f(z)| > \exp(|z|^c)$.

Proposition 28.1. *If f is an entire function of finite genus μ, then f is of finite order $\lambda \leqslant \mu + 1$.* †

If f does not have finite order, then f has infinite order and $\lambda = \infty$. For examples, $\exp(\exp(z))$ has infinite order and if g is a polynomial of degree n, then $\exp(g)$ has order n.

For the entire function f and $r > 0$, let $M(r) = M_f(r) = \max\{|f(z)| : |z| = r\}$. If f is not constant, then for r sufficiently large, $M_f(r) > 1$.

Proposition 28.2. *If f is a nonconstant entire function, then the order of f is given by*

$$\sigma = \limsup_{r\to\infty} \frac{\log(\log(M(r)))}{\log(r)}.$$

†

Proof. Suppose f has order $\lambda < \infty$ and let $\epsilon > 0$ be given. There is an $R > 0$ such that for $|z| > R$, we have $\log(|f(z)|) \leqslant |z|^{\lambda+\epsilon}$. Thus, for $r > R$,

$$\log(\log(|M(r)|) \leqslant (\lambda + \epsilon) \log(r).$$

Hence $\sigma \leqslant \lambda + \epsilon$ and thus $\sigma \leqslant \lambda$.

On the other hand, if $\sigma < \infty$, then by the definition of lim sup, given $\epsilon > 0$ there is an $R \geqslant 1$ such that if $|z| > R$, then $\frac{\log(\log(M(r)))}{\log(r)} \leqslant \sigma + \epsilon$. It follows that

$$M(r) \leqslant \exp(r^{\sigma+\epsilon})$$

for all $r > R$. Hence $\lambda \leqslant \sigma + \epsilon$ and therefore $\lambda \leqslant \sigma$. □

Problem 28.1. Suppose (a_j) is a sequence of non-zero numbers, p is a nonnegative integer and $p_1 \geqslant \rho > 0$. Show, if

$$\sum_{j=1}^{\infty} |a_j|^{-\rho} < \infty,$$

then the canonical product

$$P(z) = \prod_{j=1}^{\infty} E_p(\frac{z}{a_j})$$

has finite order $\lambda \leqslant \rho$.

29. Hadamard Factorization

Theorem 29.1. *If f is an entire of finite order λ, then f has finite genus $\mu \leqslant \lambda$.*

Before proving Theorem 29.1, we collect some consequences.

Theorem 29.2. *If f is a non-constant entire function of finite order, then f assumes each complex number with at most one exception. Moreover, if f does omit a value, then the order of f is an integer.*

Proof. Suppose f is entire of finite order λ and the range of f omits the point x. It follows that $f - x$ is an entire function of finite order λ and f never vanishes. In particular, by Corollary 7.10, there is an entire function g such that $f - x = e^g$. Since $f - x$ has finite order, by Theorem 29.1, f has finite genus and therefore g is a polynomial. It now follows that the order of f is an integer. Since g is a polynomial, it is either constant or g assumes every value (by the fundamental theorem of algebra). Hence either f is constant or $f - x$ takes every value except of course 0 so that f takes every value except x. ☐

Theorem 29.3. *If f is entire of finite order λ and if λ is not an integer, then f assumes each value infinitely often.*

Proof. Suppose f has finitely many zeros so that the standard form for f is

$$f(z) = \prod_{j=1}^{n} (z - a_j) \exp(g(z)),$$

for some n, complex numbers a_j and entire function g. By Theorem 29.1, g is a polynomial. On the other hand, the order of f and $\exp(g)$ are the same and, since the order of $\exp(g)$ is an integer, so is the order of f. Thus if the order of f is not an integer, then f is zero infinitely often.

Fixing $x \in \mathbb{C}$ and applying what has already been proved to $f - x$ it follows that f takes the value x infinitely often. ☐

29.1. **Proof of Theorem 29.1.**

Proposition 29.4. *Suppose f is entire, $f(0) = 1$ with zeros $(a_j)_{j=1}^{\infty}$ arranged in increasing order of modulus. Let g denote the lograthmic derivative of f; i.e., $g = \frac{f'}{f}$. If f has finite order λ and $p > \lambda - 1$, then the p-th derivative of g has the representation,*

$$g^{(p)}(z) = -p! \sum_{j=1}^{\infty} (a_j - z)^{-(p+1)}, \quad f(z) \neq 0.$$

†

Recall the definitions of $M(r)$ and $n(r)$.

Lemma 29.5. *For $z \in \mathbb{C}$,*

$$\lim_{r\to\infty} \sum_{j=1}^{n(r)} \left[\frac{\overline{a_j}}{r^2 - -\overline{a_j} z}\right]^{p+1} = 0. \tag{17}$$

†

Proof. For $r > 2|z|$ and $j \leqslant n(r)$,

$$|r^2 - -\overline{a_j} z| \geqslant \frac{1}{2} r^2$$

since $|a_j| < r$. Hence,

$$\left[\frac{|\overline{a_j}|}{|r^2 - -\overline{a_j} z|}\right]^{p+1} \leqslant \left(\frac{2}{r}\right)^{p+1}.$$

Using $\log(2)\, n(r) \leqslant M(2r)$ (see Problem 26.1), the sum in equation (17) is bounded by $\frac{M(2r)}{\log(2)} \left(\frac{2}{r}\right)^{p+1}$. Recall f has order $\lambda < p+1$. Choose $\epsilon > 0$ such that $\lambda + \epsilon < p + 1$. Proposition 28.2 implies

$$\log(M(2r)) \left(\frac{2}{r}\right)^{p+1} \leqslant (2r)^{\lambda+\epsilon} \left(\frac{2}{r}\right)^{p+1} \leqslant 4^{p+1} r^{-\epsilon}$$

for r sufficiently large. The conclusion of the lemma now follows. □

Lemma 29.6. *For $z \in \mathbb{C}$,*

$$\lim_{r\to\infty} \int_0^{2\pi} re^{it} \left(re^{it} - z\right)^{-(p+2)} \log(|f(re^{it})|)\, dt = 0.$$

†

Proof. Choose $\epsilon > 0$ such that $\mu = \lambda + \epsilon < p + 1$. By Proposition 28.2, for r sufficiently large $\log(|f(re^{it})|) \leqslant r^{\lambda+\epsilon}$. Thus for r large,

$$\begin{aligned}\left|re^{it} \left(re^{it} - z\right)^{-(p+2)} \log(|f(re^{it})|)\right| \leqslant & r| \left(re^{it} - z\right)^{-(p+2)} | \log(M(r)) \\ \leqslant & r^{\lambda+\epsilon+1} | \left(re^{it} - z\right)^{-(p+2)} | \approx r^{-\epsilon}.\end{aligned}$$

□

Lemma 29.7. *Suppose U is open. If $f : U \to \mathbb{C}$ is analytic and never zero, then, with $u(z) = \log(|f(z)|)$,*

$$\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{f'}{f}.$$

†

Proof of Proposition 29.4. For $h = \frac{r(z-a)}{r^2-\overline{a}z}$, verify that

$$\frac{h'(z)}{h(z)} = \frac{1}{z-a} + \frac{\overline{a}}{r^2-\overline{a}z}.$$

For $f(z) \neq 0$ and $r > |z|$, use Theorem 26.1, an application of Lemma 29.7 along with differentiating under the integral sign to obtain

$$g(z) := \frac{f'(z)}{f(z)} = \sum_{j=1}^{n(r)} \left[\frac{1}{z-a_j} + \frac{\overline{a_j}}{r^2-\overline{a_j}z}\right] + \frac{1}{2\pi}\int_0^{2\pi} \frac{2re^{it}}{(re^{it}-z)^2} \log(|f(e^{it})|)\, dt.$$

Differentiating this last equality p times gives,

$$g^{(p)}(z) = -\,p!\sum_{j=1}^{n(r)} \left[(\frac{1}{z-a_j})^{p+1} + (\frac{\overline{a_j}}{r^2-\overline{a_j}z})^{p+1}\right] + (p+1)!\frac{1}{2\pi}\int_0^{2\pi} \frac{2re^{it}}{(re^{it}-z)^{p+2}} \log(|f(e^{it})|)\, dt.$$

By Lemmas 29.5 and 29.6 the last two terms on the right hand side tend to 0 with r. □

Proposition 29.8. *Suppose f is an entire function with non-zero zeros $a_1, a_2, \dots$. If f has finite order λ, then for each $\sigma > \lambda$,*

$$\sum_{j=1}^{\infty} |a_j|^{-\sigma} < \infty.$$

In particular, f has finite rank at most λ. †

Lemma 29.9. *If f is entire, has finite order λ, m is a nonnegative integer and $f(z) = z^m g(z)$, where $g(0) \neq 0$, then the order of g is the same as the order of f.* †

Proof of Proposition 29.8. By Lemma 29.9, without loss of generality $f(0) = 1$. Fix $1 > \epsilon > 0$ and $\tau > 0$. There is no harm in assuming $0 < |a_1| \leqslant |a_2| \leqslant \dots$. Recall, from Problem 26.1, that $\log(2)\, n(r) \leqslant M(2r)$. Since f has order λ, there is an $R > 0$ such that $\log(M(2r)) \leqslant (2r)^{\lambda+\epsilon}$ for $r > R$ by Proposition 28.2. Thus, for $r > R$,

$$n(r) \leqslant \frac{(2r)^{\lambda+\epsilon}}{\log(2)} \leqslant r^{\lambda+\epsilon}\, 2^{\lambda+2}.$$

Using $k \leqslant n(|a_k|)$, it follows that there is a K such that for $k \geqslant K$,

$$k \leqslant n(2|a_k|) \leqslant (2|a_k|)^{\lambda+\epsilon} 2^{\lambda+2} \leqslant |a_k|^{\lambda+\epsilon} 4^{\lambda+2}.$$

Thus, for $k \geqslant K$,

$$4^{(1+\tau)(\lambda+2)}\, k^{-(1+\tau)} \geqslant |a_k|^{-(1+\tau)(\lambda+\epsilon)}.$$

Since the series $\sum k^c$ converges for $c < -1$, the result now follows by choosing ϵ and τ such that $(1+\tau)(\lambda+\epsilon) < \sigma$ and applying the comparison test. $\square$

Proof of Theorem 29.1. By Proposition 29.8, if f has finite order, then f has finite rank at most $\tau \leqslant \lambda$. Hence, there is an integer $\lambda \geqslant p > \lambda - 1$ with $p \geqslant \tau$. Without loss of generality assume $f(0) = 1$. Express (using Theorem 17.12) f in standard form

$$f(z) = e^{g(z)}\, P(z),$$

where $P(z) = \prod E_\tau(\frac{z}{a_j})$ and g is an entire function. If z is not a zero of f, then

$$h = \frac{f'}{f}(z) = \frac{P'}{P}(z) + g'(z).$$

Differentiating p times and applying Proposition 29.4 to h gives,

$$-p!\sum_{j=1}^{\infty}(a_j - z)^{-(p+1)} = Q^{(p)}(z) + g^{(p+1)}(z)$$

where $Q(z) = P'(z)P^{-1}(z)$. On the other hand, for a given a, and $F(z) = E_\tau(\frac{z}{a}) = (1-\frac{z}{a})e^{\psi(z)}$, where ψ is a polynomial of degree τ,

$$\frac{F'}{F}(z) = \frac{1}{a-z} + \psi' z.$$

Thus,

$$Q^{(p)}(z) = -p!\sum_{j=1}^{\infty}(a_j - z)^{-(p+1)}.$$

Therefore $g^{(p+1)} = 0$ and hence g is a polynomial of degree at most $p \leqslant \lambda$. The conclusion follows. $\square$

29.2. **The exponent of convergence and further results.** The *exponent of convergence* of a sequence (a_j) of nonzero complex numbers is

$$\rho = \inf\{c : \sum |a_j|^{-c} < \infty\}.$$

In this language Proposition 29.8 says if f has order λ, then the exponent of convergence of the non-zero zeros of f is at most λ.

Problem 29.1. Show the P defined as in Problem 28.1 has order equal to the exponent of convergence of the sequence of its non-zero zeros.

Problem 29.2. Suppose g is a polynomial of degree n and P is as in Problem 28.1. Show the order of $f = \exp(g(z))P(z)$ is the maximum of the degree of g and the order of P. (Suggestion: Use Proposition 29.8 and Problem 27.1.)

Problem 29.3. Suppose f_1, f_2 are entire of finite orders λ_1, λ_2 respectively. Show, if $\lambda_1 \neq \lambda_2$, then the product $f_1 f_2$ has order $\lambda = \max\{\lambda_1, \lambda_2\}$.

Show, by example, the conclusion can fail in the case $\lambda_1 = \lambda_2$.

30. Bloch's Theorem

Theorem 30.1. *Suppose $\Omega \subset \mathbb{C}$ is open and contains $\overline{\mathbb{D}}$. If $f : \Omega \to \mathbb{C}$ is analytic, $f(0) = 0$ and $f'(0) = 1$, then there exists and a disk $D \subset \mathbb{D}$ such that $f|_D$ is one-one and $f(D)$ contains a disk of radius $\frac{1}{72}$.*

Corollary 30.2. *Suppose $R > 0$. If f is analytic in an open set Ω containing the closure of $B(0;R)$, then $f(B(0;R))$ contains a disk of radius $\frac{1}{72}R|f'(0)|$.* †

30.1. Proof of Theorem 30.1.

Lemma 30.3. *If $f : \mathbb{D} \to \mathbb{C}$ is analytic, $f(0) = 0$ and $f'(0) = 1$, then $M = \sup\{|f(z)| : z \in \mathbb{D}\} \geqslant 1$ and*

$$B(0; \frac{1}{6M}) \subset \text{range}(f).$$

†

Proof. The Schwarz Lemma (Lemma 12.5) implies $M \geqslant 1$. Express f as the power series $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$. For $0 < r < 1$ and $j \geqslant 1$, Cauchy's estimate gives $|a_j| \leqslant Mr^{-n}$. Hence $|a_j| \leqslant M$. Thus, for $|z| = (4M)^{-1}$,

$$\begin{aligned}
|f(z)| \geqslant & |z| - \sum_{j=2}^{\infty} |a_j||z|^j \\
\geqslant & (4M)^{-1} - \sum_{j=2}^{\infty} M(4M)^{-j} \\
\geqslant & (4M)^{-1} - 16M^3 \sum_{j=0}^{\infty} (4M)^{-j} \\
= & (4M)^{-1} - \frac{1}{16M} 1 - \frac{1}{4M} \\
= & \frac{1}{4} \frac{4M-2}{M(4M-1)} \\
\geqslant & \frac{1}{6M}.
\end{aligned}$$

□

Lemma 30.4. *Let $M, R > 0$ be given. If $f : B(0;R) \to \overline{B(0,M)}$ is analytic, $f(0) = 0$ and $f'(0) \neq 0$, then*

$$B\big(0; \frac{R^2|f'(0)|^2}{6M}\big) \subset f(B(0;R)).$$

†

Proof. Define $g : \mathbb{D} \to \mathbb{C}$ by

$$g(z) = \frac{f(Rz)}{Rf'(0)}$$

and verify that g defines the hypothesis of Lemma 30.3. □

Lemma 30.5. *Suppose $R > 0$, $a \in \mathbb{C}$ and $f : B(a; R) \to \mathbb{C}$ is analytic. If*

$$|f'(z) - f'(a)| < |f'(a)|$$

for $a \neq z \in B(a; R)$, then f is one-one. †

Proof. Fix $z \neq w \in B(a; R)$, let $\gamma(t) = tz + (1-t)w$ and estimate

$$\begin{aligned}
|f(z) - f(w)| =& \left| \int_0^1 f'(\gamma(t))[z-w]\, dt \right| \\
\geqslant& \left| \int_0^1 f'(a)[z-w]\, dt \right| - \left| \int_0^1 [f'(\gamma(t)) - f'(a)][z-w]\, dt \right| \\
=& |z-w| \left[|f'(a)| - \left| \int_0^1 [f'(\gamma(t)) - f'(a)]\, dt \right| \right] \\
\geqslant& |z-w| \left[|f'(a)| - \int_0^1 |f'(\gamma(t)) - f'(a)|\, dt \right] \\
=& |z-w| \int_0^1 [|f'(a)| - |f'(\gamma(t)) - f'(a)|]\, dt > 0.
\end{aligned}$$

□

Lemma 30.6. *Suppose $\rho > 0$, $a \in \mathbb{C}$ and let $D = B(a; \frac{\rho}{3})$. If $f : B(a; \rho) \to \mathbb{C}$ is analytic and*

(i) $|f'(a)| = \frac{1}{2\rho}$; and
(ii) $|f'(z)| < \frac{1}{\rho}$ for $z \in B(a; \rho)$,

then $f|_D$ is one-one and $f(D)$ contains a disk of radius $\frac{1}{72}$. †

Proof. First observe if $g : B(a; r) \to B(0; R)$ is analytic and $g(a) = 0$, then h, defined by $h(z) = \frac{g(\frac{z}{r} + a)}{R}$ is an analytic function $h : \mathbb{D} \to \mathbb{D}$ with $h(0) = 0$. Hence, by the Schwarz Lemma 12.5,

$$|h(z)| \leqslant |z|, \ |z| < 1.$$

It follows that

$$|g(w)| = R\,|h(\frac{z-a}{r})| \leqslant \frac{R|z-a|}{r}.$$

Combining items (i) and (ii) gives

$$|f'(z) - f'(a)| < \frac{3}{2\rho}.$$

Applying the version of the Schwarz lemma above with $g = f' - f'(a)$, $r = \rho$ and $R = \frac{3}{2\rho}$ gives

$$|f'(z) - f'(a)| \leqslant \frac{3|z-a|}{2\rho^2}.$$

Thus, for $z \in D$,

$$|f'(z) - f'(a)| < \frac{1}{2\rho} = |f'(a)|.$$

An application of Lemma 30.5 shows $f|_D$ is one-one.

To complete the proof, without loss of generality assume $a = 0$. For $z \in D$, and with $g(z) = f(z) - f(a)$,

$$|g(z)| = |z|\,|\int_0^1 f'(tz)\,dt|.$$

Using item (ii),

$$|g(z)| \leqslant \frac{|z|}{\rho} < \frac{1}{3}.$$

By Lemma 30.4 with $R = \frac{\rho}{3}$ and $M = \frac{1}{3}$ gives,

$$g(D) \supset B(0; \frac{\rho^2|f'(0)|^2}{18}) = B(0; \frac{1}{72}).$$

Hence $f(D)$ contains the disk of radius $\frac{1}{72}$ centered to $f(a)$. □

Remark 30.7. For an entire function f, the function $M(r) = M_f(r)$ is increasing and continuous. ◇

Proof of Theorem 30.1. Define $h : [0,1] \to \mathbb{R}$ by $h(r) = (1-r)M_{f'}(r)$. Thus h is continuous, $h(0) = 1$ and $h(1) = 0$. By continuity of h (see Lemma 30.7) the set $h^{-1}(\{1\})$ is closed and hence contains its sup, r. In particular, $h(r) = 1$ and $h(s) < 1$ for $s > r$. Choose $|a| = r$ such that $|f'(a)| = M_{f'}(r)$. Hence,

$$|f'(a)| = \frac{h(r)}{1-r} = \frac{1}{1-r}.$$

Let $\rho = \frac{1}{2}(1-r)$. Thus $|f'(a)| = \frac{1}{2\rho}$. Further, if $|z-a| < \rho$, then

$$|z| < \frac{1}{2}(1-r) \leqslant \frac{1}{2}(1+r) < 1.$$

Hence, as $\frac{1}{2}(1+r) > r$ (and again using Lemma 30.7),

$$\begin{aligned}
|f'(z)| &\leqslant M_{f'}(|z|) \\
&\leqslant M_{f'}(\frac{1}{2}(1+r)) \\
&= h(\frac{1}{2}(1+r))(1 - \frac{1}{2}(1+r))^{-1} \\
&< \frac{1}{1-\frac{1}{2}(1+r)} = \frac{1}{\rho}.
\end{aligned}$$

An application of Lemma 30.6 completes the proof. □

31. Picard's little theorem

Theorem 31.1 (The Little Picard Theorem). *If f is entire and omits two values, then f is constant.*

Theorem 31.2 (Schottky's Theorem). *For each $0 \leqslant \beta \leqslant 1$ there a constant C_β such that if*

(i) f analytic in a simply connected set containing $\overline{\mathbb{D}}$;
(ii) f omits the values 0 and 1;
(iii) and $|f(0)| \leqslant 1$,

then $|f(z)| \leqslant C_\beta$ for $|z| < \beta$.

The following Corollary to Theorem 31.2 will be used in the proof of the Picard's Great Theorem discussed in the next section.

Corollary 31.3. *For each $0 \leqslant \beta \leqslant 1$ and $R > 0$, there is a constant C_β such that if*

(i) f is analytic on a simply connected region containing $\overline{B(0;R)}$;
(ii) f omits the values 0 and 1; and
(iii) $|f(0)| \leqslant 1$,

then $|f(z)| \leqslant C_\beta$ for $|z| \leqslant \beta R$. †

Proof. Apply Theorem 31.2 to the function $g(z) = f(Rz)$. □

Aside from several problems at the end, the rest of this section is devoted to proving Theorems 31.1 and 31.2.

Lemma 31.4. *If $x, y, z \in \mathbb{C}\backslash\{0\}$ and $x^2 = z$ and $y^2 = z - 1$, then*

$$\frac{1}{4}[(x+y) + \frac{1}{x+y}]^2 = z$$

and

$$(x+y)^2 + (x+y)^{-2} = 4z - 2.$$

†

Proof. Let $S = x + y$ and compute,

$$\begin{aligned}
\frac{1}{4}\big[S + \frac{1}{S}\big]^2 =& \frac{1}{4}\frac{(S^2+1)^2}{S^2} \\
=& \frac{1}{4}\frac{(x^2+y^2-2xy+1)^2}{(x^2+y^2-2xy)} \\
=& \frac{1}{4}\frac{(2z-2xy)^2}{2h-1-2xy} \\
=& \frac{1}{4}\frac{4z^2+4z^2-4z-8zxy}{2z-1-2xy} \\
=& z.
\end{aligned}$$

The second part of the lemma follows immediately from the first. □

Let
$$L = \{\pm(\log(\sqrt{n} - \sqrt{n-1})) + \frac{ik\pi}{2} : n \in \mathbb{N}^+,\ k \in \mathbb{Z}\}.$$

Lemma 31.5. *If f is analytic on a simply connected domain and if the range of f contains neither 0 nor 1, then there is an analytic function $g : B(0;R) \to \mathbb{C}$ such that*

(i)
$$f = -\exp(i\pi\cosh(2g)).$$

(ii) the range of f is disjoint from L; and
(iii) the set L intersects every disk of radius two.

†

Proof of Lemma 31.5 and Theorem 31.2. Let G denote the simply connected domain of f. Since f omits the value 0, by Corollary 7.10(iv), there is an analytic function h on G such that $f = \exp(2\pi ih)$. Without loss of generality, we may assume $0 \leqslant \operatorname{Re} h(0) < 1$. Since f omits the value 1, the range of h contains no integers. In particular, there are analytic functions ψ_j on G such that
$$h(z) = \exp(\psi_0(z)), \quad h(z) - 1 = \exp(\psi_1(z)).$$
Let $S_j(z) = \exp(\frac{1}{2}\psi_j(z))$ so that $S_0^2 = h$ and $S_1^2 = h - 1$. The function $S = S_0 - S_1$ does not take the value 0. Hence there is an analytic function g on G such that $S = \exp(g)$. Without loss of generality, assume $0 \leqslant \operatorname{Im} g(0) < 2\pi$. Compute, using Lemma 31.4,
$$\begin{aligned}\cosh(2g) + 1 &= \frac{\exp(2g) + \exp(-2g)}{2} + 1\\ &= \frac{1}{2}\big[\exp(g) + \exp(-g)\big]^2\\ &= \frac{1}{2}\big[S + \frac{1}{S}\big]^2\\ &= 2h.\end{aligned}$$

Thus,
$$f(z) = \exp(\pi i[\cosh(2g) + 1]) = -\exp(\pi i\cosh(2g)).$$
At this point the proof of item (i) of Lemma 31.5 is complete.

To prove item (ii), arguing the contrapositive, suppose there is a point $y \in \Omega \subset \mathbb{C}$ and $n \in \mathbb{N}^+$ and $k \in \mathbb{Z}$ such that
$$g(y) = \pm(\log(\sqrt{n} + \sqrt{n-1})) + \frac{ik\pi}{2}.$$
In this case, and taking frequent advantage of the identity $(\sqrt{n} + \sqrt{n-1})^{-1} = \sqrt{n} - \sqrt{n-1}$,
$$\exp(\pm 2g(y)) = \pm(\sqrt{n} + \sqrt{n-1})^{\mp 2}.$$

Hence, using Lemma 31.4,

$$\cosh(2g(y)) = \frac{\pm}{2}\big[(\sqrt{n}+\sqrt{n-1})^{\mp 2} + (\sqrt{n}+\sqrt{n-1})^{\pm 2}\big] = \pm(2n-1).$$

Hence,

$$f(y) = -\exp(\pm \pi i(2n-1)) = 1.$$

Turning to item (iii), fix a point $z = a+bi$ with $a,b \in \mathbb{R}$. There is a $k \in \mathbb{Z}$ such that $\frac{ik\pi}{2} \leqslant b \leqslant \frac{i(k+1)\pi}{2}$. Likewise, assuming $a \geqslant 0$, there is an $n \in \mathbb{N}^+$ such that $\log(\sqrt{n}+\sqrt{n-1}) \leqslant a \leqslant \log(\sqrt{n+1}+\sqrt{n})$. Now,

$$\log(\sqrt{n+1}+\sqrt{n}) - \log(\sqrt{n}+\sqrt{n-1}) \leqslant 1.$$

Hence, the distance from z to L is at most $\sqrt{1+(\frac{\pi}{2})^2} \leqslant 2$. The proof of Lemma 31.5 is now complete.

To prove Theorem 31.2, first observe that, by rotating, it can be assumed that $f(0)$ is real and $0 \leqslant f(0) \leqslant 1$. For now, suppose in fact $\frac{1}{2} \leqslant f(0) \leqslant 1$. In this case, $\log(2) \geqslant \log(|f(0)|) \geqslant 0$ and

$$\log(|f(0)|) = \log(\exp(2\pi \operatorname{Im} h(0))) = 2\pi \operatorname{Re} h(0).$$

Thus,

$$|h(0)| \leqslant |\operatorname{Re} h(0)| + |\operatorname{Im} h(0)| \leqslant 1 + \frac{\log(2)}{2\pi}.$$

Let $C_0 = 1 + \frac{\log(2)}{2\pi}$.

Next,

$$\begin{aligned}\big|S_0(0) \pm S_1(0)\big| \leqslant & |S_0(0)| + |S_1(0)| \\ = & |h(0)|^{\frac{1}{2}} + |h(0)-1|^{\frac{1}{2}} \\ \leqslant & C_0^{\frac{1}{2}} + (C_0+1)^{\frac{1}{2}}.\end{aligned}$$

Let $C_1 = C_0^{\frac{1}{2}} + (C_0+1)^{\frac{1}{2}}$.

If $|S(0)| \geqslant 1$, then

$$\begin{aligned}|g(0)| \leqslant & |\operatorname{Re} g(0)| + |\operatorname{Im} g(0)| \\ & \leqslant \log(|S(0)|) + 2\pi \leqslant \log(C_1) + 2\pi.\end{aligned}$$

If $|S(0)| \leqslant 1$, then $\log(|S(0)|) < 0$ and

$$\begin{aligned}|g(0)| \leqslant & |\operatorname{Re} g(0)| + |\operatorname{Im} g(0)| \\ & \leqslant -\log(|S(0)|) + 2\pi \\ & = \log(\frac{1}{S(0)} + 2\pi \\ & = \log(S_0(0) + S_1(0)) + 2\pi \leqslant \log(C_1) + 2\pi.\end{aligned}$$

Hence in any event, $|g(0)| \leqslant C_2 := \log(C_1) + 2\pi$.

For $a \in \mathbb{D}$, Corollary 30.2 says the set $g(B(a; 1-|a|))$ contains a disk of radius $\frac{(1-|a|)|g'(a)|}{72}$. On the other hand, items (ii) and (iii) imply that the range of g contains no disk of radius two. Hence, for $|a| < 1$,

$$|g'(a)| \leqslant \frac{72}{1-|a|}.$$

It follows that

$$\begin{aligned} |g(a)| \leqslant & |g(0)| + |g(a) - g(0)| \\ \leqslant & C_2 + \int_0^1 |g'(ta)||a|\, dt \\ \leqslant & C_2 + \frac{72|a|}{1-|a|}. \end{aligned}$$

Thus if $|z| \leqslant \beta < 1$, then

$$|g(a)| \leqslant C_2 + \frac{72\beta}{1-\beta}$$

and, in view of the relation between f and g, the proof is complete in the case $\frac{1}{2}f(0) \leqslant 1$; i.e., there is a constant D_β such that $|f(z)| \leqslant D_\beta$ for $|z| \leqslant \beta$.

Finally, suppose $0 \leqslant f(0) \leqslant \frac{1}{2}$. In this case $\tilde{f} = 1 - f$ satisfies $\frac{1}{2}\tilde{f}(0) \leqslant 1$ and consequently,

$$|1 - f(z)| = |\tilde{f}(z)| \leqslant C_2 + \frac{72\beta}{1-\beta}.$$

Choosing $C_\beta = 1 + D_\beta$ completes the proof. $\square$

Proof of Theorem 31.1. Suppose f omits the values $y \neq w$ and fix $R > 0$. Let $F(z) = \frac{f(z)-y}{w-y}$ omits the values $0, 1$. Hence, by Lemma 31.5, there is an entire g such that $F = -\exp(i\pi 2g)$ such that the range of g is disjoint from L. Hence the range of g contains no disk of radius two by Lemma 31.5(iii). If g is not constant, then there is a point $p \in \mathbb{C}$ such that $g'(p) \neq 0$. From Corollary 30.2, $g(B(p; R))$ contains a disk of radius $\frac{1}{72}R|g'(p)|$, a contradiction. $\square$

31.1. Problems.

Problem 31.1. Show, a nonconstant meromrphic function omits at most three values in $\mathbb{C}_\infty$.

Problem 31.2. Show, if $n \geqslant 3$ is an integer, then $f^n + g^n = 1$ has no nontrivial entire solutions.

Show, if f, g are entire and $f^2 + g^2 = 1$, then there is an entire function h such that $f = \cos(h)$ and $g = \sin(h)$.

32. The Montel-Caratheodory Theorem

Theorem 32.1. *Let $\Omega \subset \mathbb{C}$ be an open connected set. The*

$$\mathscr{F} = \{f : \Omega \to \mathbb{C} : f \text{ is analytic and omits the values } 0, 1\}$$

is a normal family. In particular, if (f_n) is a sequence from $\mathscr{F}$, then there is a subsequence (g_k) of (f_n) such that (g_n) converges uniformly on compact sets to ∞ or to some analytic function $g : \Omega \to \mathbb{C}$.

Proof. Fix a point $b \in \Omega$ and let

$$\begin{aligned}\mathscr{F}_1 =& \{f \in \mathscr{F} : |f(b)| \leqslant 1\}\\ \mathscr{F}_2 =& \{f \in \mathscr{F} : |f(b)| \geqslant 1\}.\end{aligned}$$

The first step is to use Theorem 14.4 (Montel's Theorem) to very that $\mathscr{F}_1$ is normal. Accordingly it suffices to show that $\mathscr{F}_1$ is locally bounded. Evidently the set of points S in Ω for which $\mathscr{F}_1$ is locally bounded is an open set. To prove S is closed, suppose (z_n) is a sequence from S that converges to $z_0 \in \Omega$. There is an $r > 0$ such that $B(z_0; r) \subset \Omega$. There is an N so that $|z_N - z_0| < \frac{r}{2}$. Since $\mathscr{F}_1$ is bounded at z_N, there is a C such that $|f(z_N)| \leqslant C_1$ for all $f \in \mathscr{F}_1$. An application of Schottky's Theorem (Theorem 31.2) to the ball $B(z_N; \frac{r}{2})$ gives a constant C_2 such that $|f(z)| \leqslant C_2 C_1$ for all $f \in \mathscr{F}_1$ and $z \in B(z_N; \frac{r}{2})$. Since S is a nonempty subset of Ω that is both open and closed, $S = \Omega$.

Observe, if $f \in \mathscr{F}_2$, then, as f never vanishes, $\frac{1}{f} \in \mathscr{F}$. In particular, as $\mathscr{F}_1$ is normal, if (f_n) is a sequence from $\mathscr{F}_2$, then, by passing to a subsequence if necessary, there is an analytic function h on Ω such that $(\frac{1}{f_n})$ converges uniformly on compact sets to h. By Theorem 14.2, since noe of the $\frac{1}{f_n}$ are zero, either h never vanishes of h is identically zero. Now verify, if h is identically 0, then (f_n) converges to ∞ uniformly on compact sets; and otherwise (f_n) converges uniformly to $\frac{1}{h}$ on compact sets. Hence $\mathscr{F}_2$ is normal.

An easy argument based on normality of $\mathscr{F}_j$ for $j = 1, 2$ completes the proof. □

33. The Great Picard Theorem

Compate the following Theorem to Theorem 9.6.

Theorem 33.1 (The Great Picard Theorem)**.** *If $\Omega \subset \mathbb{C}$ is open, $y \in \Omega$, $f : \Omega \backslash \{y\} \to \mathbb{C}$ is analytic and has an essential singularity at y, then f assumes every value, with at most one exception, infinitely often.*

Corollary 33.2. *If f is entire and not a polynomial, then, with at most one exception, f assumes every value infinitely often.* †

Given a polynomial p with n distinct zeros. The function $f(z) = p(z)\exp(z)$ takes every value except 0 infinitely often and the value 0 exactly n times. Hence, there is no general statement about the number of times the possible exceptional value is assumed.

Proof. If f has a pole at infinity, then f is a polynomial by Problem 9.4. □

Proof of Theorem 33.1. Arguing the contrapositive, suppose f omits two values. Without loss of generality, assume

(i) $\Omega = B(0;R)$ (for some $R > 0$);
(ii) f has an essential singularity at 0;
(iii) f omits the values 0 and 1.

For notational ease, let $U = B(0;R)\backslash\{0\}$. Define $f_n : U \to \mathbb{C}$ by $f_n(z) = f(\frac{z}{n})$. Thus each f_n omits the values 0 and 1 and therefore (f_n) is a sequence from the normal family of Theorem 32.1 (Montel-Caratheodory). Hence there is a subsequence (g_k) of (f_n) such that (g_n) converges uniformly on compact sets to either ∞ or to some analytic function $g : U \to \mathbb{C}$. In the case of convergence to ∞, the function f has a pole at 0.

Suppose $(g_k = f_{n_k})$ converges uniformly on copmpact sets to an analytic function g. The function g is bounded, by some M on $\{|z| = \frac{R}{2}$. Hence $|f_{n_k}(z)| \leqslant M + 1$ on $|z| = \frac{R}{2n_k}$ for k sufficiently large. Using the maximum modulus theorem, it follows that f is bounded the annuli $\{r < |z| < \frac{R}{2}\}$ for each $0 < r$ sufficiently small. Hence f has a removable singularity at 0.

□

33.1. **Problems.**

Problem 33.1. Show that $\sin(z) = z$ has infinitely many solutions.

Problem 33.2. Show $f(z) = e^z - z$ takes every value infinitely often.

34. Harmonic Conjugates

Recall a function u on an open set $\Omega \subset \mathbb{C} = \mathbb{R}^2$ is *harmonic* if it has continuous second partials (denoted $u \in C^2$) and satisfies Laplace's equation (9). Assuming only that $u \in C^2$, define

$$\partial u = u_x - iu_y.$$

Remark 34.1. If u has a harmonic conjugate v (and locally it does), then $f = u + iv$ is analytic and, by the Cauchy-Riemann equations,

$$\partial u = u_x + iv_x = f'.$$

◇

Proposition 34.2. *Suppose $\Omega \subset \mathbb{C}$ is open and $u : \Omega \to \mathbb{R}$ is C^2. The function u is harmonic if and only if $f = \partial u$ is analytic. In this case, if γ is a closed rectifiable curve in Ω, then*

$$\int_\gamma \partial u = i \int_\gamma (u_x dy - u_y dx).$$

†

In the context of Proposition 34.2,

$$^*du = u_x dy - u_y dx.$$

is the *conjugate differential* of u. If u has a harmonic conjugate v, then

$$dv = v_x dx + v_y dy = {}^*du.$$

Theorem 34.3. *Suppose $\Omega \subset \mathbb{C}$ is an open set and $u : \Omega \to \mathbb{R}$ is C^2. The following are equivalent.*

(i) u has a harmonic conjugate in Ω;
(ii) ∂u has a primitive;
(iii) $^*du$ is an exact differential (in Ω); and
(iv) for each closed rectifiable curve in Ω,

$$\int_\gamma {}^*du = 0.$$

34.1. Proofs.

Proof of Proposition 34.2. Assuming the domain of γ is $[0,1]$, writing $\gamma(t) = a(t) + ib(t)$ and using exactness of $u_x dx + u_y dy$ and closedness of γ,

$$\begin{aligned}
\int_\gamma i^*du =& \frac{1}{2}\int_0^1 i[u_x(\gamma(t))b'(t) - u_y(\gamma(t))a'(t)]\,dt \\
=& \int_0^1 i[u_x(\gamma(t))b'(t) - u_y(\gamma(t))a'(t)]\,dt + \int_0^1 [u_x(\gamma(t))a'(t) + u_y(\gamma(t))b'(t)]\,dt \\
=& \int_0^1 [u_x(\gamma(t)) - iu_y(\gamma(t))]\,[a'(t) + ib'(t)]\,dt \\
=& \int_\gamma \partial u.
\end{aligned}$$

□

Proof of Theorem 34.3. Assuming (i), there is a harmonic function v such that $g = u + iv$ is analytic. Hence from the Cauchy-Riemann equations,

$$g' = u_x + iv_x = u_x - iu_y = 2\partial u.$$

Conversely, if there is an analytic $g = u + iv$ such that $g' = 2\partial u$, then from the equation above $v_x = -u_y$. On the other hand, $g' = -iu_y + v_y$ and hence $v_y = u_x$. Hence v is a harmonic conjugate of u.

A discussion earlier in this section shows (i) implies (iii). The converse is easily seen to be true.

It is a standard fact from calculus that (iii) implies (iv). On the other hand, condition (iv) implies (ii) from Corollary 2.2 (a version of Cauchy's Theorem). □

35. Gauss's Theorem and Green's Formula

In these notes, a *nice* domain Ω of genus g is an open subset of $\mathbb{C}$ bounded by $g+1$ smooth (continuously differentiable) closed non-intersecting curves $\Gamma = (\Gamma_0, \Gamma_1, \dots, \Gamma_g)$. Further, we assume that $\mathbb{C}\backslash\Omega = K_0 \cup_{j=1}^{g} K_j$ where the K_j are compact domains, K_0 is the unbounded component and Γ_j is the boundary of K_j. Often for simplicity Γ_0 is taken as the unit circle and all the other curves are assumed to lie in the disk. The curve(s) Γ is assumed oriented (so we view Γ as a (union of) curve(s)) so that the region lies to the left. A *nice multiply connected domain* is a domain of genus g for some g.

The following theorem is a standard fact from multivariable calculus.

Theorem 35.1 (Gauss's Theorem). *Suppose Ω is a nice multiply connected domain with boundary Γ. If p, q are defined in a an open set containing the closure of Ω and have continuous first partials, then*

$$\iint\limits_{\Omega} [p_x + q_y] \, dA = \int_{\Gamma} p dy - q dx.$$

(The left hand side is the integral against the area element $dA = dx\, dy$ and the left the line integral over the (oriented) curve Γ.)

Suppose $u : \Omega \subset \mathbb{C} \to \mathbb{R}$ is C^1 and $\gamma(s) = (x(s), y(s))$ is a C^1 path in Ω parameterized by arclength s. For $f(s) = u(\gamma(s))$, by the chain rule

$$f'(s) = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} = \nabla u \cdot T,$$

where T is the tangent vector to the curve at s. Note that this derivative really only depends upon T, and not the particular curve γ. Accordingly, define the *tangential derivative* of u at $p = \gamma(s)$ by

$$\frac{\partial u}{\partial s} = \nabla u \cdot T.$$

(The partial notation reflects the idea that we are differentiating in the tangential direction The *normal derivative* or *derivative with respect to the outward normal* is, by definition

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial y}\frac{dx}{ds} - \frac{\partial u}{\partial x}\frac{dy}{ds} = \nabla u \cdot N,$$

where $N = (b, -a)$ is the normal vector. In particular, $\frac{\partial u}{\partial n} ds = u_y dx - u_x dy$.

Define the *Laplacian*

$$\triangle u = u_{xx} + v_{yy}.$$

Thus u is harmonic if and only if $\triangle u = 0$.

Theorem 35.2 (Green's Formula). *If Ω is a nice multiply connected domain and u, v are defined and C^2 in a neighborhood of the closure of Ω, then*

$$\iint\limits_{\Omega} (u \triangle v - v \triangle u) dA = \int_{\Gamma} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds. \tag{18}$$

Proof. Let $p = u\frac{\partial v}{\partial x}$ and $q = u\frac{\partial v}{\partial y}$ and observe,

$$p_x = uv_{xx} + u_x v_x, \quad q_y = uv_{yy} + u_y v_y.$$

By Theorem 35.1,

$$\iint_\Omega [uv_{xx} + u_x v_x + uv_{yy} + u_y v_y]\, dA = \int_\Gamma [uv_x dy - uv_y dx].$$

Hence,

$$\iint_\Omega u\triangle v dA + \iint_\Omega [u_x v_x + u_y v_y]\, dA = \int_\Gamma u[v_x dy - v_y dx] = \int_\Gamma u\frac{\partial v}{\partial n} ds. \tag{19}$$

Using the analogous equation obtained by switching the roles of u and v and taking the difference yields formula (18). □

36. The periods of a harmonic function

Sheldon Axler, in an article in the American Math Monthly, refers to the following result as the logarithmic conjugation theorem and champions it as a way to avoid talking about the periods of a harmonic function.

Suppose Ω is a nice multiply connected domain of genus g with complement $\cup_{j=0}^g K_j$ (with K_0 being the unbounded component). A set $\gamma = \{\gamma_1, \dots, \gamma_g\}$ of smooth curves in Ω is a is a *basis* if $n_{\gamma_j}(w) = 1$ for $w \in K_j$ and $n_{\gamma_j}(w) = 0$ for $w \in K_k$ for $k \neq j$.

Theorem 36.1. *Suppose Ω is a nice multiply connected domain of genus g with complement $\cup_{j=1}^g K_j$ and $\gamma = \{\gamma_1, \dots, \gamma_g\}$ is a basis for Ω. Given a harmonic function $u : \Omega \to \mathbb{C}$, let*

$$c_j = \frac{1}{2\pi i}\int_{\gamma_j} \partial u = \frac{1}{2\pi}\int_{\gamma_j} {}^*du \tag{20}$$

The numbers c_j are real and independent of the choice of basis γ and for any choice a_j of points from the interior of K_j for $1 \leqslant j \leqslant g$, there is an analytic function $f : \Omega \to \mathbb{C}$ such that

$$u(z) = \operatorname{Re} f(z) + \sum_{j=1}^g c_j \log(|z - a_j|). \tag{21}$$

Finally, if the partials of u extend continuously to the boundary, then so does f; and if f extends continuously to the boundary, then so does u and the formula of (21).

The c_j are the *periods* of u about the boundary component Γ_j.

Remark 36.2. The idea behind the proof is the following. Fix a base point $b \in \Omega$. It is possible to define locally an analytic function whose real part is u by choosing a path γ from b to z and considering, somewhat informally,

$$\int_b^z \partial u := \int_\gamma \partial u + u(b).$$

The difficulty is that different paths from b to z can lead to different values. On the other hand, these values differ in a predictable fashion, namely up to some $\mathbb{Z}$ linear combination of the periods, by the Cauchy integral formula. Our forefathers referred to these as *multiple-valued functions*. The proof of theorem then amounts to *correcting the periods*, in this case using the harmonic functions (with well understood periods) $\log(|z-a_j|)$. $\diamond$

The proof of Theorem 36.1 will use the following lemma.

Lemma 36.3. *Given a point $a \in \mathbb{C}$,*

$$\partial \log(|z-a|) = \frac{1}{z-a}.$$

$\dagger$

Proof. Given a point z, choose a branch of the logarithm $\log(z-a)$ in a neighborhood N of z. In particular, $u = \operatorname{Re}\log(z-a)$ and thus the imaginary part of $\log(z-a)$ is a harmonic conjugate of u. The lemma now follows by Remark 34.1 $\square$

Proof of Theorem 36.1. Choose curves γ_j in Ω such that $n_{\gamma_j}(w) = 1$ for $w \in K_j$ and $n_{\gamma_j}(w) = 0$ for $w \in K_i$ for $i \neq j$. Let

$$c_j = \frac{1}{2\pi i}\int_{\gamma_j} \partial u.$$

Since

$$\begin{aligned}\operatorname{Im} c_j =& \frac{-1}{2\pi}\operatorname{Re}\int_{\gamma_j} \partial u \\ =& \frac{-1}{2\pi}\operatorname{Re}\int_{\gamma_j} [u_x - iu_y](dx+idy) \\ =& \frac{-1}{2\pi}[u_x dx + u_y dy] = 0\end{aligned}$$

since the expression is the integral of an exact differential over a closed curve. Thus c_j is real. Let

$$g(z) = [\partial u - \sum_{j=1}^{g} \frac{c_j}{z-a_j}].$$

Now suppose σ is a closed curve in Ω with winding number n_j about K_j (meaning $n_\sigma(w) = n_j$ for each $w \in K_j$). It follows that γ is homologous to $\gamma = \sum_{j=1}^{g} m_j\gamma_j$; i.e., $\sigma - \gamma$ is homologous to 0. Hence, by Cauchy's integral formula,

$$\begin{aligned}\int_\sigma g =& \int_\gamma g \\ =& n_j \sum_j \int_{\gamma_j} \partial u - \sum c_j n_{\gamma_j}(a_j) = 0\end{aligned} \tag{22}$$

by the choice of the c_j. It follows, from Corollary 16.3, that g has a primitive f; i.e., there is an analytic function $f : \Omega \to \mathbb{C}$ such that $f' = g$. Indeed, a choice of f is obtained by fixing a point $b \in \Omega$ and defining $f(z) = \int_b^z g$ from which it will follow that f extends continuously to the boundary if the partials of u do.

Let

$$v(z) = \operatorname{Re} f(z) + \sum_{j=1}^{g} c_j \log(|z - a_j|).$$

Let $h(z) = \operatorname{Re} f(z)$ and note that $h_x = \operatorname{Re} f'$. Likewise, let $h_j(z) = \log(|z - a_j|)$ and observe, by Lemma 36.3,

$$(h_j)_x = \operatorname{Re} \partial h_j = \operatorname{Re} \frac{1}{z - a_j}.$$

Thus

$$v_x = \operatorname{Re}[f' + \sum_{j=1}^{g} c_j \partial h_j] = \operatorname{Re} \partial u = u_x.$$

A similar argument shows $v_y = u_y$. Hence u and v agree up to a constant. Adjusting f by this constant gives formula (21). □

37. The reflection principle for harmonic functions

An *analytic closed curve* γ is the image of the unit circle $\mathbb{T}$ under a mapping that is bianalytic in an open set containing $\mathbb{T}$. The nice multiply connected domain Ω has *analytic boundary* if each of the boundary components Γ_j is a simple analytic closed curve. In this case we may assume that the bianalytic mappings defining Γ_j map the inside of the unit circle into Ω.

Theorem 37.1. *Suppose the boundary of Ω consists of simple closed analytic arcs. If $u : \overline{\Omega} \to \mathbb{R}$ is continuous, harmonic on Ω and $u|_{\Gamma_j}$ is constant for $0 \leqslant j \leqslant g$, then u extends to a harmonic function on an open set containing $\overline{\Omega}$.*

The proof of Theorem 37.1 occupies the rest of this section.

Lemma 37.2. *Suppose $\rho > 0$ and let*

$$\mathbb{H} = \{z : 0 < \operatorname{Im} z < \rho\}, \quad \mathbb{H}^+ = \{z : 0 \leqslant \operatorname{Im} z < \rho\}.$$

If

(1) $u : \mathbb{H}^+ \to \mathbb{R}$ is continuous;
(2) $u|_{\mathbb{H}}$ is harmonic; and
(3) $u|_{\{\operatorname{Im} z = 0\}} = 0$,

then u extends uniquely to a harmonic function on $\mathbb{S} = \{-\rho < \operatorname{Im} z < \rho\}$. Moreover, if u is 2π periodic, then so is its extension. †

Proof. Define $v : \mathbb{S} \to \mathbb{R}$ by

$$v(z) = \begin{cases} u(z) & z \in \mathbb{H}^+ \\ -u(\overline{z}) & -\rho < \operatorname{Im} z < 0. \end{cases}$$

Thus v extends u and is continuous in view of items (1) and (3). Moreover, v is harmonic in $\mathbb{H}$ by hypotheses from which it follows by a simple calculation that v is harmonic on the set $\{-\rho < 0 < \operatorname{Im} z\}$. By Theorem 22.6, it is enough to show that for each point $z \in \mathbb{S}$ there is an $R > 0$ such that v satisfies the maximum principle in $B(z; R) \subset \mathbb{S}$. Hence, it suffices to show that if $\operatorname{Im} z = 0$, $r > 0$ and $\overline{B(z; r)} \subset \mathbb{S}$, then

$$\int_0^{2\pi} v(z + re^{it})dt = v(z) = 0,$$

a conclusion that follows from the symmetry $v(z + re^{-it}) = v(\overline{z + r^{it}}) = -v(z + r^{it})$.

Finally, if u is 2π periodic, then, from its definition, so is v. □

Lemma 37.3. *Fix $r > 0$ and consider the annular regions*

$$\mathbb{A} = \{r < |z| < 1\}, \quad \mathbb{A}^+ = \{r < |z| \leqslant 1\}.$$

If

(i) $u : \mathbb{A}^+ \to \mathbb{R}$ is continuous;
(ii) $u|_{\mathbb{A}}$ is harmonic; and
(iii) $u = 0$ on $\mathbb{T}$,

then u extends to a harmonic function on an open set containing $\mathbb{A}^+$. †

Proof. Let $\rho = -\log(r) > 0$ and let $\mathbb{S}$ denote the strip $\{-\rho < \Im z < \rho\}$. In particular, $e(z) = \exp(iz)$ maps $\mathbb{S}$ conformally onto the annular region $\mathscr{A} = \{r < |z| < \frac{1}{r}\}$. Define $\mathbb{H}$ and $\mathbb{H}^+$ as in Lemma 37.2 and defined $\tilde{u} : \mathbb{H}^+ \to \mathbb{R}$ by $\tilde{u} = u \circ e$. It follows that $\tilde{u}$ is continuous, 2π periodic and $\tilde{u}|_{\mathbb{H}}$ is a harmonic. Hence, by Lemma 37.2, $\tilde{u}$ extends to a 2π periodic harmonic function $\tilde{v}$ on all of $\mathbb{S}$. Since $\tilde{v}$ is 2π periodic, there is a harmonic function v on $\mathscr{A}$ such that $\tilde{v} = v \circ e$. (Compare with Lemma 20.6.) Indeed, given $z \in \mathscr{A}$, simply observe, that if ℓ_1 and ℓ_2 are any two branches of the log defined in a neighborhood of z, the $\ell_1(\zeta) = \ell_2(\zeta) + 2\pi ik$ for some integer k. Hence, $v(z) = \tilde{v}(i\ell_j(z))$ is well defined. It is harmonic as it is the composition of a harmonic function with (locally) an analytic function. Now $v \circ e = \tilde{v}$ and thus $v \circ e|_{\mathbb{H}^+} = \tilde{u} = u \circ e$. Hence $v|_{\mathbb{A}^+} = u$.

□

Lemma 37.4. *Let Ω be a nice multiply connected domain with boundary $\Gamma = \cup \Gamma_j$. Fix k. If*

(1) u is continuous on $\Omega \cup \Gamma_k$;
(2) Γ_k is an simple closed analytic curve;
(3) u is constant on Γ_k,

then there is an open set $\Omega' \supset \Omega \cup \Gamma_k$ and a harmonic function on Ω' extending u. †

Theorem 37.5 (Jordan Curve Theorem)**.** *If γ is a simple closed curve in $\mathbb{C}$, then the open set $\mathbb{C}\backslash\{\gamma\}$ consists of two components.*

The unbounded component is the *outside* of $\{\gamma\}$. The bounded component is the *inside* of $\{\gamma\}$.

Proof of Lemma 37.4. Assume, without loss of generality, that $u = 0$ on Γ_k. By hypothesis, there is an annular region $\mathscr{A} = \{r < |z| < \frac{1}{r}\}$ and an analytic function $f : \mathscr{A} \to \mathbb{C}$ bianalytic on to its range such that Γ_k is the image of $\mathbb{T}$. Without loss of generality we may assume this mapping takes $\mathbb{A} = \{r < |z| < 1\}$ into Ω; and $\mathbb{A}' = \{1 < |z| < \frac{1}{r}\}$ into the complement of $\overline{\Omega}$. The composition $u \circ f$ defined on $\mathbb{A}^+ = \{r < |z| \leqslant 1\}$ satisfies the hypotheses Lemma 37.3. Hence $u \circ f$ extends to a harmonic function $\hat{v}$ on $\mathscr{A}$. Letting $v = \hat{v} \circ f^{-1}$ completes the proof. □

Proof of Theorem 37.1. The hypotheses allow the application of Lemma 37.4 to each boundary component. □

38. Harmonic measure, the period matrix and the Abel-Jacobi map

A nice multiply connected domain Ω of genus g with boundary Γ is a Dirichlet domain. Let w_j denote the solution to the Dirichlet problem on Ω with boundary values $w_j|_{\Gamma_i} = 1$ if $i = j$ and 0 if $i \neq j$. The harmonic functions $w_0, w_1, \ldots, w_g$ are *harmonic measure* for Ω. This terminology is of course rather outdated. Let, for $\{\gamma_1, \ldots, \gamma_g\}$ a basis for Ω,

$$p_{j,k} = \frac{1}{2\pi} \int_{\gamma_j} \frac{\partial w_k}{\partial n} ds = \int_{\Gamma} \frac{\partial w_k}{\partial n} ds.$$

Thus $P_{j,k}$ is the period of w_k about Γ_j. Note that Green's Theorem implies $P_{k,j} = P_{j,k}$ and thus $P = P^T$. The matrix $P = (p_{j,k})$ is the *period matrix* for Ω.

For the remainder of this section, assume Γ consists of simple closed analytic curves.

Proposition 38.1. *The period matrix is symmetric and positive definite.* †

Proof. The harmonic measures w_j extend harmonically across the boundary by Theorem 37.1. Given real numbers $c_1, \ldots, c_g$ not all zero, let $w = \sum c_j w_j$. By the primitive version of Greens' formula applied to $u = w = v$, equation (19),

$$\begin{aligned} 0 < \iint_{\Omega} (w_x^2 + w_y^2)\, dA &= \sum_{j,k=1}^{g} c_j c_k \int_{\Gamma} w_j \frac{\partial w_k}{\partial n} ds \\ &= \sum_{j,k} P_{j,k}\, c_j\, c_k. \end{aligned}$$

It follows, letting $c \in \mathbb{R}^g$ denote the vector with entries c_j, that $c^T P c > 0$ whenever $c \neq 0$. Since also P is symmetric, it is positive definite. □

Let $\mathbb{Z}^g$ denote column vectors with entries from $\mathbb{Z}$ and let $\mathbb{L}$ denote the lattice

$$\mathbb{L} = \mathbb{Z}^g + iP\mathbb{Z}^g = \{m + iPn : m, n \in \mathbb{Z}^g\},$$

where P is the period matrix for Ω. The Jacobian variety of Ω is the quotient space $\mathbb{C}^g/\mathbb{L}$. Let $w_1, \dots, w_g$ denote the harmonic measures for Ω, let

$$w = \begin{pmatrix} w_1 \\ \vdots \\ w_g \end{pmatrix}$$

and define ∂w in the obvious fashion.

Proposition 38.2. *Fix a point $b \in \Omega$. If γ is a closed (rectifiable) path in Ω, then*

$$\frac{1}{2\pi}\int_\gamma \partial w \in iP\mathbb{Z}^g.$$

Thus the mapping $f : \Omega \to \mathbb{C}^g$ defined by

$$f(z) = \frac{1}{\pi}\int_b^z \frac{1}{2}\partial w,$$

where the integral is taken over any path from b to z in Ω, is well defined and analytic. †

Proof. The curve γ is homologous to $\sum m_j \Gamma_j$ for some choice of integers m_j. Hence,

$$\frac{1}{2\pi}\int_\gamma \partial w_k = i\sum P_{j,k} m_j$$

and therefore

$$\frac{1}{2\pi}\int_\gamma \partial w = iPm.$$

It follows that f is well defined. It is locally analytic and hence analytic.

Finally, □

39. Appendix A

Lemma 39.1. *Suppose* $\Omega \subset \mathbb{C}$ *is an open set,* $\gamma : [a,b] \to \Omega$ *is a rectifiable path. If* $\mathcal{F}$ *is an equicontinuous family offunctions* $f : \Omega \to \mathbb{C}$*, then for every* $\epsilon > 0$ *there exists a polygonal path* $\Gamma : [a,b] \to \Omega$ *such that* $\Gamma(a) = \gamma(a)$ *and* $\Gamma(b) = \gamma(b)$ *and, for each* $f \in \mathcal{F}$*,*

$$\left| \int_\gamma f - \int_\Gamma f \right| < \epsilon.$$

†

Proof. Observe that the proof in Conway works at this level of generality □

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