On Generators of the Hardy and the Bergman Spaces

Joseph A. Cima
Department of Mathematics the University of North Carolina at Chapel Hill
Joint work with Valentin V. Andreev, Lamar University and Miron B. Bekker, Department of Mathematics the University of Pittsburgh at Johnstown

Introduction

It is well known that any (closed) subspace of the Hardy space $H^2(\mathbb{D})$ or the Bergman space $A^2(\mathbb{D})$ which is invariant under multiplication by z (the $z-$invariant subspace) is also an invariant subspace of any bounded analytic Toeplitz operator. The converse is not in general true.

In this talk we consider analytic Toeplitz operators M_φ of multiplication by φ on the Hardy space $H^2(\mathbb{D})$ and the Bergman space $A^2(\mathbb{D})$ with the property that the function which is a constant (e.g. $f(z) \equiv 1$) is a cyclic vector for such operators. Such a function φ we call a generator. We characterize such generators in terms of $M_\varphi-$ invariant subspaces which are also $z-$invariant.

The problem of determining whether a given bounded analytic function in the the unit disk $\mathbb{D}$ is a generator is very hard. Some important results in this direction in the case of Hardy space were obtained in the paper by B. M. Solomyak [9].

In [7] and [8] Donald Sarason introduced a notion of a weak* generator of H^∞. We show that each weak* generator is a generator in our sense. If ψ is a weak* generator of H^∞ then $\psi-$invariant subspaces have some additional property, see Theorem 7 below.

Weak* generators often occur in the process of studying wandering property of invariant subspaces of analytic Toeplitz operators. We also consider this question for the case when the symbol of the Toeplitz operator is a generator, see Theorem 3.

The condition that $H^\infty \cap \mathfrak{M}$ is dense in the invariant subspace $\mathfrak{M}$ plays the crucial role in our work. In Section 3 we study this condition in the case of invariant subspaces of the Hardy space $H^2(\mathbb{D})$ and the Bergman space $A^2(\mathbb{D})$.

Generators

In what follows $\mathbb{D}$ denotes the unit disk and $\mathfrak{H}$ denotes either Hardy space $H^2(\mathbb{D})$ or Bergman space $A^2(\mathbb{D})$.

Definition 1. A function $\varphi \in H^\infty(\mathbb{D})$ is called a generator for $\mathfrak{H}$ if polynomials in φ are dense in $\mathfrak{H}$, i.e.

$$\overline{l.h.\{\varphi^n : n = 0, 1, \ldots\}} = \mathfrak{H} \tag{1}$$

where *l.h.* means linear hull and bar means closure in the norm of $\mathfrak{H}$.

Condition (1) implies that a vector $\mathbf{1}$ (the constant function 1) is a cyclic vector for the analytic Toeplitz operator M_φ of multiplication by φ.

Since in a locally convex space the weak closure of a subspace coincides with its norm closure, a function φ is a generator if polynomials in φ are weakly dense in $\mathfrak{H}$.

Clearly, φ is univalent. Indeed, if $\varphi(z_1) = \varphi(z_2)$, $z_1, z_2 \in \mathbb{D}$, $z_1 \neq z_2$, then $P(\varphi(z_1)) = P(\varphi(z_2))$ for any polynomial P. Consequently, $f(z_1) = f(z_2)$ for any $f \in \overline{l.h.\{\varphi^n : n = 0, 1, \ldots\}}$ which contradicts (1).

A notion of a generator is closely related to the notion of weak* generator introduced by D. Sarason in [7] and [8].
A function $\psi \in H^\infty$ is called a weak* generator of H^∞ if polynomials in ψ are dense in the weak-star topology of H^∞.

Theorem 1. *1. Every weak* generator is a generator in the sense of Definition 1.*

2. There exist a bounded univalent function φ which is a generator in the sense of Definition 1 but is not a weak generator.*

Proof. *1.* Recall that the space H^∞ is the dual to the quotient space L^1/H^1_0, $H^\infty = (L^1/H^1_0)^*$. A local basis of the weak* topology on H^∞ at $h_0 \in H^\infty$ is formed by the following sets

$$\left\{h \in H^\infty : \left|\frac{1}{2\pi}\int_0^{2\pi}[h(e^{it}) - h_0(e^{it})]g_j(e^{it})dt\right| < \epsilon, j = 1, 2, \ldots, n, g_j \in L^1\right\} \tag{2}$$

Let ψ be a weak* generator of H^∞. Then any set of the form (2) contains a polynomial in ψ for any $h_0 \in H^\infty$. Taking $g_j = \bar{f}_j$,

$f_j \in H^1$, we obtain that for any $h_0 \in H^\infty(\mathbb{D})$, for any $\epsilon > 0$, and for any $f_j \in H^1(\mathbb{D})$, $j = 1, 2, \ldots, n$, there is a polynomial $P(\psi)$ in ψ such that

$$|\frac{1}{2\pi} \int\limits_0^{2\pi} \left[P(\psi(e^{it})) - h_0(e^{it})\right] \overline{f_j(e^{it})} dt| < \epsilon, \quad j = 1.2. \ldots, n. \tag{3}$$

Since $H^2(\mathbb{D}) \subset H^1(\mathbb{D})$ we take $f_j \in H^2(\mathbb{D})$ and (3) gives

$$| < P(\psi) - h_0, f_j >_{H^2} | < \epsilon,$$

that is the weak closure of polynomials in ψ contains all H^∞. In particular that closure contains all polynomials in z. Therefore weak closure of polynomials in ψ coincides with $H^2(\mathbb{D})$, hence the closure of polynomials in ψ in $H^2(\mathbb{D})$-norm is $H^2(\mathbb{D})$. Consequently any weak* generator of $H^\infty(\mathbb{D})$ is a generator of $H^2(\mathbb{D})$.

Remark 1. Using similar arguments one can show that if ψ is a weak* generator of H^∞, then the of the set of polynomials in ψ is dense in H^p for any $1 < p < \infty$.

2. Let φ be a bounded univalent function and put $G = \varphi(\mathbb{D})$. It is easily seen that the function φ is a generator of $H^2(\mathbb{D})$ if and only if the polynomials are dense in $H^2(G)$ in the norm of $H^2(G)$. In [1] J. Akeroyd constructed a bounded simply connected domain G such that the polynomials are dense in $H^2(G)$. At the same time the mapping function φ is not a weak* generator of $H^\infty(\mathbb{D})$. This statement follows from the fact that Akeroyd's domain is a subset of the unit disk $\mathbb{D}$ and the following result of D. Sarason (see [8], Corollary 2, page 527):
The function φ fails to be a weak generator if there is a domain B containing G properly such that*

$$\sup_{z \in B} |f(z)| = \sup_{z \in G} |f(z)|.$$

In the case of the Bergman space we note that the space $(A^1(\mathbb{D}))^*$ properly contains $H^\infty(\mathbb{D})$. Let τ be the topology on $H^\infty(\mathbb{D})$ defined by the family of seminorms

$$p_f(\psi) = |\int_{\mathbb{D}} \psi(z)\overline{f(\zeta)}d\sigma(z)|, \qquad \psi \in H^\infty(\mathbb{D}), \quad f \in A^1(\mathbb{D}), \quad (4)$$

where σ is the normalized Lebesque measure of the unit disk $\mathbb{D}$. A function $\psi \in H^\infty(\mathbb{D})$ is called a τ-generator of $H^\infty(\mathbb{D})$ if polynomials in ψ are dense in H^∞ in $\tau-$topology. Clearly the topology τ is the Bergman space version of the weak* generator of $H^\infty(\mathbb{D})$ and a $\tau-$generator of H^∞ is the Bergman space version of a weak* generator of $H^\infty(\mathbb{D})$.

Applying the same arguments as above one concludes that every $\tau-$ generator of H^∞ is a generator of the Bergman space $A^2(\mathbb{D})$ in the sense of Definition 1. Suppose now that φ is a generator of $A^2(\mathbb{D})$ in the sense of Definition 1. Therefore polynomials in φ are dense in $A^2(\mathbb{D})$ in the Bergman norm. In particular, polynomials in φ are dense in $A^2(\mathbb{D})$ in the weak topology. This means that for any $\epsilon > 0$ and any $h_0 \in H^\infty$, and for any finite set of functions $f_j \in A^2(\mathbb{D})$ there is a polynomial P such that

$$| \int_{\mathbb{D}} (P(\varphi(z)) - h(z)) \overline{f_j(z)} d\sigma(z)| < \epsilon.$$

Since $A^1(\mathbb{D})$ contains functions that do not belong to $A^2(\mathbb{D})$, φ is not a $\tau-$generator of $H^\infty(\mathbb{D})$.

A (closed) subspace of $\mathfrak{H}$ is called φ-invariant if it is invariant under the operator M_φ. We also denote by $Lat_{\mathfrak{H}}(\varphi)$ the lattice of φ-invariant subspaces of $\mathfrak{H}$.

For a set $S \subset \mathfrak{H}$ we denote $[S]_\varphi$ the smallest $\varphi-$ invariant subspace containing S. Similarly, $[S]_z$ is the smallest $z-$invariant subspace containing S.

The next theorem gives a characterization of a generator in terms of its invariant subspaces.

Theorem 2. *Let φ be a bounded univalent function in the unit disk $\mathbb{D}$. In order for φ to be a generator it is necessary that every $\varphi-$invariant subspace $\mathfrak{M} \subset \mathfrak{H}$ such that $\overline{\mathfrak{M} \cap H^\infty} = \mathfrak{M}$ is also $z-$invariant, and sufficient that the invariant subspace $[\mathbf{1}]_\varphi$ be $z-$invariant.*

Proof. Suppose φ is a generator, $f \in \mathfrak{M} \cap H^\infty$ and $h \in \mathfrak{M}^\perp$. Since $\mathfrak{M}$ is $\varphi-$invariant, $P(\varphi)f \in \mathfrak{M}$ for any polynomial P, that is

$$< P(\varphi)f, h >= 0.$$

The left side of the last equality can be written as $< P(\varphi), \bar{f}h >=< P(\varphi), T_{\bar{f}}h >$, where $T_{\bar{f}}$ is a Toeplitz operator on $\mathfrak{H}$. Since $f \in H^\infty$ the operator $T_{\bar{f}}$ is bounded and $T_{\bar{f}}h \in \mathfrak{H}$. Thus we have

$$0 =< P(\varphi)f, h >=< P(\varphi), T_{\bar{f}}h > .$$

Now pick $\epsilon > 0$. Because φ is a generator there is a polynomial P such that

$$| < z - P(\varphi), T_{\bar{f}}h > | < \epsilon.$$

Since $\epsilon > 0$ is arbitrary it follows that

$$0 =< z, T_{\bar{f}}h >=< M_z f, h > .$$

Hence $M_z f \in \mathfrak{M}$ for any $f \in \mathfrak{M} \cap H^\infty$ and, because this intersection is dense in $\mathfrak{M}$ one concludes that $M_z \mathfrak{M} \subset \mathfrak{M}$.

To prove the converse statement consider the $\varphi-$invariant subspace $[\mathbf{1}]_\varphi$. Since this subspace is the closure in $\mathfrak{H}$ of polynomials in φ, functions from H^∞ are dense in it. Consequently, $[\mathbf{1}]_\varphi$ is also $z-$invariant. But any $z-$invariant subspace of $\mathfrak{H}$ which contains the function $\mathbf{1}$ coincides with the whole $\mathfrak{H}$. Therefore $[\mathbf{1}]_\varphi = \mathfrak{H}$, that is φ is a generator. □

Remark 2. It was proved earlier that a bounded univalent function ψ is a weak* generator of H^∞ if and only if $Lat_{\mathfrak{H}}(\psi) = Lat_{\mathfrak{H}}(z)$. For the case $\mathfrak{H} = H^2(\mathbb{D})$ it was proved by D. Sarason in [7], for $\mathfrak{H} = A^2(\mathbb{D})$ the statement was proved by P. Bourdon in [3].

Corollary 1. *Let φ be a generator of $\mathfrak{H}$ and let $\mathfrak{M}$ be a $\varphi-$invariant subspace. If* $\dim \mathfrak{M}^{\perp} < \infty$ *then the subspace $\mathfrak{M}$ is z-invariant.*

The corollary follows from the fact that H^{∞} is dense in $\mathfrak{H}$ and the statement below that was proved for the more general situation in [6], Lemma 2.1:
Let a Banach space X be decomposed as the direct sum of a subspace Y and a finite-dimensional subspace Z:

$$X = Y \dot{+} Z,$$

and L is a dense linear subset of X. Then $Y \cap L$ is dense in Y.

The following statement was proved in [4], Lemma 4.1.

Lemma 1. *Let* φ *be a bounded univalent function on the unit disk* $\mathbb{D}$ *with* $\varphi(0) = 0$. *If* $\mathfrak{M}$ *is a* $z-$*invariant subspace of* $\mathfrak{H}$, *then* $\mathfrak{M} \ominus M_z\mathfrak{M} = \mathfrak{M} \ominus M_\varphi\mathfrak{M}$.

Theorem 3. *Let* φ *be a bounded univalent function on the unit disk* $\mathbb{D}$ *with* $\varphi(0) = 0$. *Assume that* φ *is a generator of* $\mathfrak{H}$ *and* $\mathfrak{M}$ *is a* $\varphi-$ *invariant subspace of* $\mathfrak{H}$ *such that:*

(a) $\mathfrak{M} \cap H^\infty(\mathbb{D})$ *is dense in* $\mathfrak{M}$;

(b) $(\mathfrak{M} \ominus \varphi\mathfrak{M}) \cap H^\infty(\mathbb{D})$ *is dense in* $\mathfrak{M} \ominus \varphi\mathfrak{M}$.

Then

$$\mathfrak{M} = [\mathfrak{M} \ominus \varphi\mathfrak{M}]_\varphi\,. \tag{5}$$

Proof. From Theorem 2 it follows that the subspace $\mathfrak{M}$ is M_z-invariant and Lemma 1 gives that $\mathfrak{M} \ominus M_z\mathfrak{M} = \mathfrak{M} \ominus M_\varphi\mathfrak{M}$. If $\mathfrak{H} = H^2(\mathbb{D})$ the fact that $\mathfrak{M} = [\mathfrak{M} \ominus M_z\mathfrak{M}]_z$ is the well known Wold decomposition. If $\mathfrak{H} = A^2(\mathbb{D})$ then from the result of Aleman, Richter, and Sundberg(see [**?**]) it follows that $\mathfrak{M}$ is the smallest $z-$invariant subspace that contains $\mathfrak{M} \ominus M_z\mathfrak{M}$.

Observe that because $\varphi \in H^\infty(\mathbb{D})$ we have $[\mathfrak{M} \ominus M_\varphi\mathfrak{M}]_\varphi \cap H^\infty(\mathbb{D})$ is dense in $[\mathfrak{M} \ominus M_\varphi\mathfrak{M}]_\varphi$. Since the last subspace is $\varphi-$invariant we refer again to Theorem 2 and conclude that it is $z-$invariant, and is a subspace of $\mathfrak{M}$. Therefore $[\mathfrak{M} \ominus M_\varphi\mathfrak{M}]_\varphi = \mathfrak{M}$. □

Remark 3. If in Theorem 3 $\dim(\mathfrak{M} \ominus \varphi\mathfrak{M}) < \infty$, then condition (a) implies condition (b).

Let $\mathfrak{M}$ be a $\varphi-$invariant subspace of $\mathfrak{H}$. In what follows the quantity $\dim(\mathfrak{M} \ominus \varphi\mathfrak{M})$ is called the $\varphi-$index of $\mathfrak{M}$. The $z-$index of a $z-$invariant subspace is defined similarly.

Theorem 4. *Let* φ *be a generator of* $\mathfrak{H}$ *and* $\varphi(0) = 0$. *Assume* $f \in (\mathfrak{H} \cap H^\infty)$ *and* $\mathfrak{M} = [f]_\varphi$. *Then the* $\varphi-$*index of* $\mathfrak{M}$ *equals 1 and* $\mathfrak{M} = [f]_z$. *Consequently, the* $z-$*index of* $\mathfrak{M}$ *is also 1.*

Proof. The proof of the first statement of the theorem is, in fact, repetition of the corresponding proof of Theorem 4, Chapter 8, of [5]. We include it for completeness.

Represent f in the form $f = f_1 + f_2$, where $f_1 \in \mathfrak{M} \ominus \varphi\mathfrak{M}$ and $f_2 \in \varphi\mathfrak{M}$. Choose an arbitrary $g \in \mathfrak{M} \ominus \varphi\mathfrak{M}$. Then $g \in [f]_\varphi$, consequently there is a sequence of polynomials of φ, say $Q_n(\varphi)$, such that $\|Q_n(\varphi)f - g\| \to 0$. Define $h_n^{(1)}$ and and $h_n^{(2)}$ by the formulas

$$h_n^{(1)} = Q_n(\varphi(0))f_1 - g \in \mathfrak{M} \ominus \varphi\mathfrak{M},$$
$$h_n^{(2)} = (Q_n(\varphi) - Q_n(\varphi(0)))f_1 + Q_n(\varphi)f_2 \in \varphi\mathfrak{M}.$$

The second inclusion follows from the fact that $Q_n(\varphi) - Q_n(\varphi(0)) = (\varphi - \varphi(0))R_n(\varphi)$, $R_n(\phi)$ is a polynomial of φ and $\varphi(0) = 0$. We have $h_n^{(1)} + h_n^{(2)} = Q_n(\varphi)f - g$, therefore

$$\|h_n^{(1)}\|^2 + \|h_n^{(2)}\|^2 = \|Q_n(\varphi)f - g\|^2 \to 0.$$

In particular, the property $\|h_n^{(1)}\| = \|Q_n(\varphi(0))f_1 - g\| \to 0$ implies $g = \lambda f_1$, and therefore the $\varphi-$index of $\mathfrak{M}$ is 1.

From our assumptions and Theorem 2 it follows that $\mathfrak{M}$ is $z-$invariant. Put $\mathfrak{M}' = [f]_z$. Then $\mathfrak{M}' \subset \mathfrak{M}$ and $\mathfrak{M}'$ is $z-$invariant. Hence $\mathfrak{M}'$ is also a $\varphi-$invariant subspace which contains f. But $\mathfrak{M} = [f]_\varphi$ is *the smallest* $\varphi-$invariant subspace which contains f. Consequently, $\mathfrak{M} \subset \mathfrak{M}'$. Hence $\mathfrak{M} = \mathfrak{M}'$.

Finally, any single-generated z-invariant subspace has $z-$index 1.
$\square$

Density of $H^\infty(\mathbb{D})$ in the Invariant Subspaces of a Generator

The previous considerations raise the following questions:
Let φ be a generator of $\mathfrak{H}$ and $\mathfrak{M}$ is a $\varphi-$invariant subspace of $\mathfrak{H}$ such that $\dim(\mathfrak{H} \ominus \mathfrak{M}) = \infty$.
Is it true that $\mathfrak{M} \cap H^\infty \neq \{0\}$?
Is $\mathfrak{M} \cap H^\infty$ dense in $\mathfrak{M}$?

For the case of $\mathfrak{H} = A^2(\mathbb{D})$ the answer is given by the following theorem.

Theorem 5. *For any function ψ analytic and bounded in the unit disk D there is a single-generated $\psi-$invariant subspace $\mathfrak{M} \subset A^2(\mathbb{D})$ such that $\mathfrak{M} \cap H^\infty(\mathbb{D}) = \{0\}$.*

Proof. Let $f \in A^2(\mathbb{D})$ be such that its zeros do not satisfy the Blaschke condition and put $\mathfrak{M} = [f]_\psi$. Clearly, zeros of any function which belong to $\mathfrak{M}$ also do not satisfy the Blaschke condition, consequently, $\mathfrak{M} \cap H^\infty(\mathbb{D}) = \{0\}$. $\square$

Now we consider the case $\mathfrak{H} = H^2(\mathbb{D})$. Let $f \in H^2(\mathbb{D}) \setminus H^\infty(\mathbb{D})$. Then a function $f_T \in H^\infty$ is called $f-$truncating, if $f_T f \in H^\infty$. There are many ways to construct an $f-$truncating function. One of them is the following. Define a real valued function $\omega(e^{it})$ as follows:

$$\omega(e^{it}) = \begin{cases} 1 & \text{if } |f(e^{it})| \leq 1 \\ 1/|f(e^{it})| & \text{if } |f(e^{it})| \geq 1. \end{cases}$$

We have $|\omega(e^{it})| \leq 1$ and $\log \omega \in L^1(\mathbb{T})$. Therefore there is an outer function $f_T \in H^\infty \subset H^2$ such that $|f_T(e^{it})| = \omega(e^{it})$. It is obvious that $f f_T \in H^\infty(\mathbb{D})$ and $\sup\{|f(z) f_T(z)| : |z| < 1\} \leq 1$.

Assume now that φ is a generator. Therefore, there is a sequence of polynomials of φ, say $\{P_n(\varphi)\}$ such that $\|P_n(\phi) - f_T\|_2 \to 0$. Such a sequence $\{P_n(\varphi)\}$ we shall call an $f-$truncating sequence. Elements of such a sequence are functions from $H^\infty(\mathbb{D})$. Since

$$\int_0^{2\pi} |P_n(\varphi(e^{it})) - f_T(e^{it})|^2 dt \to 0$$

there is a subsequence $P_{n_k}(\varphi)$ such that $P_{n_k}(\varphi(e^{it})) \to f_T(e^{it})$ for a.e. $t \in [0, 2\pi]$. Without loss of generality we may assume that $\{P_n(\varphi)\}$ converges to f_T almost everywhere on $\mathbb{T}$. In particular $P_n(\varphi(e^{it}))f(e^{it}) \to f_T(e^{it})f(e^{it})$ on the set of full measure on $\mathbb{T}$.

Theorem 6. *Let φ be a generator and let $\mathfrak{M}$ be a $\varphi-$invariant subspace of the Hardy space $H^2(\mathbb{D})$. Suppose $f \in \mathfrak{M}$ is not in $H^\infty(\mathbb{D})$ and an f-truncating sequence $\{P_n(\varphi)\}$ is uniformly bounded in H^∞-norm, that is* $\sup_n \|P_n(\varphi)\|_\infty \leq C < \infty$. *Then $\mathfrak{M} \cap H^\infty(\mathbb{D}) \neq \{0\}$.*

Proof. Let $f_T \in H^2 \cap H^\infty$ be the $f-$ truncating function constructed in the previous paragraph. We need to show that $P_n(\varphi)f \to f_T f$ in H^2-norm. Since $P_n(\varphi)f \in \mathfrak{M}$ this will prove the statement. We have

$$\int_0^{2\pi} |P_n(\varphi(e^{it}))f(e^{it}) - f_T(e^{it})f(e^{it})|^2 dt =$$
$$\int_0^{2\pi} |P_n(\varphi(e^{it})) - f_T(e^{it})|^2 |f(e^{it})|^2 dt.$$

The expression under integral sign converges to zero almost everywhere and is dominated by $C|f|^2$ where C is a positive constant. Now Lebesgue's dominated convergence theorem gives the desired result. $\square$

Theorem 7. *Let ψ be a weak* generator of H^∞ and let $\mathfrak{M}$ be a $\psi-$invariant subspace of $H^2(\mathbb{D})$. Then $\mathfrak{M} \cap H^\infty$ is dense in $\mathfrak{M}$ in H^2 norm.*

Proof. Let the functions $f \in \mathfrak{M}$ and f_T be as in the previous theorem. For any $h \in H^2(\mathbb{D})$ and any polynomial P we have

$$< P(\psi)f - f_T f, h >= \frac{1}{2\pi}\int_0^{2\pi}(P(\psi(e^{it})) - f_T(e^{it}))f(e^{it})\overline{h(e^{it})}dt.$$

Now pick $\epsilon > 0$. Since $f\bar{h} \in L^1(\mathbb{T})$ and ψ is a weak-star generator of H^∞, there is a polynomial P in ψ such that the absolute value of the last integral is less than ϵ. Since $P(\psi)f \in \mathfrak{M}$ it means that $f_T f$ belongs to the weak closure of $\mathfrak{M}$. Since weak closure of a subspace coincides with its norm closure, it proves that $\mathfrak{M} \cap H^\infty \neq \{0\}$. Put $\mathfrak{M}' = c.l.h.\{P(\psi)f_T f : P$ is a polynomial$\}$. Then the subspace $\mathfrak{M}'$ has the following properties:

1. $\mathfrak{M}' \subset \mathfrak{M}$;

2. $\mathfrak{M}'$ is $\psi-$invariant;

3. $\mathfrak{M}' \cap H^\infty$ is dense in $\mathfrak{M}'$ in H^2 norm.

Let Ω is the collection of all subspaces of $\mathfrak{M}$ which have these three properties. We have shown that $\Omega \neq \emptyset$. Partially order Ω by the set inclusion. By the Hausdorff's maximal principle there exists a maximal totally ordered subcollection Ω' of Ω. Denote by $\mathfrak{M}_0$ the union of all $\mathfrak{M}'$, where $\mathfrak{M}'$ is a member of Ω'. Then $\mathfrak{M}_0$ is a maximal subspace of $\mathfrak{M}$ which satisfies the three conditions above.

We claim that $\mathfrak{M}_0 = \mathfrak{M}$. Observe first that $\mathfrak{M} \setminus \mathfrak{M}_0$ does not contain any H^∞ vectors. Suppose $\mathfrak{M}_0 \neq \mathfrak{M}$ and pick $h \in \mathfrak{M} \ominus \mathfrak{M}_0$. Then $h \notin H^\infty$. Let h_T be the h-truncating function. Then $h_T h \in H^\infty$, hence $h_T h \in \mathfrak{M}_0$ and $P(\psi) h_T h \in \mathfrak{M}_0$ for any polynomial P. Consequently,

$$\frac{1}{2\pi} \int_0^{2\pi} P(\psi(e^{it})) h_T(e^{it}) |h(e^{it})|^2 dt = 0$$

for any polynomial P. Since $h_T|h| \in L^1(\mathbb{T})$ and ψ is a weak-star generator of H^∞ from the last equality one easily deduces that

$$\frac{1}{2\pi} \int_0^{2\pi} e^{int} h_T(e^{it}) |h(e^{it})|^2 dt = 0, \qquad n = 0, 1, 2, \ldots$$

from which it follows that

$$h_T |h|^2 = k, \tag{6}$$

where $k \in H^1_0$. Therefore

$$|h|^2 = \frac{k}{h_T}.$$

Since the left side is a real-valued function and right side is a function of the Nevanlina class and $k(0) = 0$ one concludes that that equality (5) is possible only for $h = 0$ and $k = 0$. It proves our claim and the theorem. □

Combining the last theorem with Theorem 2 we obtain the following statement.

Corollary 2. *Let ψ be a bounded univalent function in the unit disk $\mathbb{D}$. Then ψ is a weak* generator of H^∞ if and only if every $\psi-$invariant subspace of $H^2(\mathbb{D})$ is also $z-$invariant.*

The statement of the Corollary was obtained by D. Sarason [8] using another method.

References

1. John Akeroyd, *Density of the polynomials in the Hardy space of certain slit domains.* Proceeding of the AMS, **15** (1992), No 4, 1013-1021.
2. A. Aleman, S. Richter, C. Sundberg, *Beurling's theorem for the Bergman space*. Acta Math. **177** (1996), 275-310.
3. P. Bourdon, *Density of the polynomials in Bergman spaces*. Pacific. J. Math, **130**, no.2 (1987), 215-221.
4. Brent J. Carswell, *Univalent mappings and Invariant Subspaces of the Bergman and Hardy Spaces*. Proc. of AMS **131**, no.4, 1233-1241.
5. P. Duren and A. Schuster, *Bergman Spaces*. American Mathematical Society, Providence, R.I., 2004.

6. I. C. Gohberg and M. G.Krein, *The basic propositions on defect numbers, root numbers and indices of linear operators*, Uspehi Mat. Nauk **12** (1957), no. 2 (74), 43-118; English transl. Amer. Math. Soc. Transl. (2) **13** (1960), 185-264.
7. D. Sarason, *Invariant subspaces and unstarred operator algebras*. Pacific J. Math, **17**, no. 3, (1966), 145-162.
8. D. Sarason, *Weak-star generators of H^∞*. Pacific J. Math, **17**, no. 3, (1966), 519-528.
9. B. M. Solomyak, *Cyclic families of functions for analytic Toeplitz operators.* Translated from Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta im. V. A. SteklovaAN SSSR, Vol. 157, pp. 88-102, 1987.