Clark Measures for Rational Inner Functions on the Polydisk

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Southeast Analysis Meeting
March 15, 2024

Clark Measures on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

Note: If μ is a positive Borel measure on $\mathbb{T} = \partial\mathbb{D}$, then

$$(P_\mu)(z) := \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} \, d\mu(\zeta) \text{ is positive and harmonic on } \mathbb{D},$$

and the converse is true (Herglotz).

Def. ϕ is inner if $\phi \in \mathrm{Hol}(\mathbb{D})$, $\phi : \mathbb{D} \to \overline{\mathbb{D}}$ and for a.e. $\zeta \in \mathbb{T}$, $\lim_{r \nearrow 1} |\phi(r\zeta)| = 1$.

Def. For ϕ a nonconstant inner function and $\alpha \in \mathbb{T}$,

$$\mathrm{Re}\left(\frac{\alpha + \phi(z)}{\alpha - \phi(z)}\right) = \frac{1 - |\phi(z)|^2}{|\alpha - \phi(z)|^2} = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} \, d\sigma_\alpha(\zeta),$$

for some positive, Borel measure σ_α. These measures $\{\sigma_\alpha : \alpha \in \mathbb{T}\}$ are called **Clark measures associated to ϕ.**

An Example

Ex. Let $\phi(z) = z^n$ and fix $\alpha \in \mathbb{T}$.

Goal: Find σ_α with $\dfrac{1-|z|^{2n}}{|\alpha - z^n|^2} = \displaystyle\int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2}\, d\sigma_\alpha(\zeta).$

- Let $z = r\tau$ with $\tau \in \mathbb{T}$ so, $\dfrac{1-r^{2n}}{|\alpha - r^n\tau^n|^2} = \displaystyle\int_{\mathbb{T}} \frac{1-r^2}{|\zeta - r\tau|^2}\, d\sigma_\alpha(\zeta).$
- Let $\zeta_1, \dots, \zeta_n$ satisfy $\zeta_j^n = \alpha$. Then $\sigma_\alpha = \sum_{k=1}^n c_k \delta_{\zeta_k}$.
- By plugging in $z = r\zeta_k$, we can solve to get $c_k = \frac{1}{n}$.

Def. A finite Blaschke product is a function of the form

$$\phi(z) = \lambda \prod_{j=1}^{n} \frac{z - a_j}{1 - \bar{a}_j z}, \qquad \text{where } a_1, \dots, a_n \in \mathbb{D} \text{ and } \lambda \in \mathbb{T}.$$

Note: These are exactly the rational, inner functions on $\mathbb{D}$.

Ex. Let $\phi(z) = \prod_{j=1}^{n} \frac{z-a_j}{1-\bar{a}_j z}$, where $a_1, \ldots, a_n \in \mathbb{D}$.

- Fix $\alpha \in \mathbb{T}$.
- Solve $\phi(\zeta) = \alpha$ to get $\zeta_1, \ldots, \zeta_n \in \mathbb{T}$.
- Then $\sigma_\alpha = \sum_{j=1}^{n} \frac{1}{|\phi'(\zeta_j)|} \delta_{\zeta_j}$.

Some Applications

Aleksandrov Disintegration Theorem

For $g \in C(\mathbb{T})$, $\int_{\mathbb{T}} g(\zeta) dm(\zeta) = \int_{\mathbb{T}} \int_{\mathbb{T}} g(\zeta) d\sigma_\alpha(\zeta) dm(\alpha)$.

Clark Theory

- Let $H^2(\mathbb{D}) = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_{H^2}^2 := \lim_{r \nearrow 1} \int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta) < \infty \right\}$.

- For ϕ inner, define the associated model space $K_\phi := H^2 \ominus \phi H^2$ and the associated compression of the shift is $S_\phi = P_{K_\phi} M_z|_{K_\phi}$.

Clark's Theorem

Let ϕ be inner with $\phi(0) = 0$. Then

- the rank-one unitary perturbations of S_ϕ are of the form $U_\alpha = S_\phi + \alpha\langle \cdot, \frac{\phi}{z} \rangle_{H^2}$.

- Each U_α is unitarily equivalent to M_z on $L^2(\sigma_\alpha)$.

- The operator $V_\alpha : K_\phi \to L^2(\sigma_\alpha)$ satisfying $M_z V_\alpha = V_\alpha U_\alpha$ is

$$V_\alpha \left(\frac{1 - \phi(z)\overline{\phi(w)}}{1 - z\bar{w}} \right) = \frac{1 - \alpha\overline{\phi(w)}}{1 - z\bar{w}}, \quad \text{for all } w \in \mathbb{D}.$$

Def. Let $\mathbb{D}^d = \{z = (z_1, \dots, z_d) \in \mathbb{C}^d : \text{each } |z_j| < 1\} = \mathbb{D} \times \cdots \times \mathbb{D}$.
$\mathbb{T}^d = \{\zeta = (\zeta_1, \dots, \zeta_d) \in \mathbb{C}^d : \text{each } |\zeta_j| = 1\} = \mathbb{T} \times \cdots \times \mathbb{T}$.

Note: If μ is a positive Borel measure on $\mathbb{T}^d$ with Fourier coefficients supported in $\mathbb{Z}_+^d \cup (-\mathbb{Z}_+)^d$, then

$$(P_\mu)(z) := \int_{\mathbb{T}^d} \left(\prod_{j=1}^{d} \frac{1 - |z_j|^2}{|\zeta_j - z_j|^2} \right) \, d\mu(\zeta),$$

is positive and pluriharmonic on $\mathbb{D}^d$ and the converse is true.

Def. ϕ is inner if $\phi \in \text{Hol}(\mathbb{D}^d)$ and for a.e. $\zeta \in \mathbb{T}^d$, $\lim_{r \nearrow 1} |\phi(r\zeta)| = 1$. For ϕ nonconstant and $\alpha \in \mathbb{T}$,

$$\text{Re}\left(\frac{\alpha + \phi(z)}{\alpha - \phi(z)} \right) = \frac{1 - |\phi(z)|^2}{|\alpha - \phi(z)|^2} = \int_{\mathbb{T}^d} \left(\prod_{j=1}^{d} \frac{1 - |z_j|^2}{|\zeta_j - z_j|^2} \right) \, d\sigma_\alpha(\zeta),$$

for some nice σ_α. These measures $\{\sigma_\alpha : \alpha \in \mathbb{T}\}$ are called **Clark measures associated to ϕ.**

An Example (E. Doubtsov)

Consider $\phi(z) = \dfrac{2z_1z_2 - z_1 - z_2}{2 - z_1 - z_2}$, which is rational, inner on $\mathbb{D}^2$.

Each σ_α is supported on $\mathcal{L}_\alpha =$ the closure of $\{\zeta \in \mathbb{T}^2 : \phi(\zeta) = \alpha\}$. To describe σ_α, describe its behavior on $f \in C(\mathbb{T}^2)$.

- $\alpha \neq -1$: If $g_\alpha(\zeta_1) = \frac{2\alpha - \alpha\zeta_1 + \zeta_1}{2\zeta_1 - 1 + \alpha}$, $\mathcal{L}_\alpha = \{(\zeta_1, g_\alpha(\zeta_1)) : \zeta_1 \in \mathbb{T}\}$ &

$$\int_{\mathbb{T}^2} f(\zeta)\, d\sigma_\alpha(\zeta) = \int_{\mathbb{T}} f(\zeta_1, g_\alpha(\zeta_1)) \frac{2|1 - \zeta_1|^2}{|2\zeta_1 - 1 + \alpha|^2}\, dm(\zeta_1).$$

- $\alpha = -1$: $\mathcal{L}_\alpha = \{(\zeta_1, 1), (1, \zeta_2) : \zeta_1, \zeta_2 \in \mathbb{T}\}$ &

$$\int_{\mathbb{T}^2} f(\zeta)\, d\sigma_\alpha(\zeta) = \frac{1}{2}\int_{\mathbb{T}} f(1, \zeta_2)\, dm(\zeta_2) + \frac{1}{2}\int_{\mathbb{T}} f(\zeta_1, 1)\, dm(\zeta_1).$$

Some Results on $\mathbb{D}^d$

Aleksandrov Disintegration Theorem (Doubtsov 2020)

For $g \in C(\mathbb{T}^d)$, $\int_{\mathbb{T}^d} g(\zeta)dm(\zeta) = \int_{\mathbb{T}} \int_{\mathbb{T}^d} g(\zeta)d\sigma_\alpha(\zeta)dm(\alpha)$.

Clark Theory:

- $H^2(\mathbb{D}^d) = \left\{ f \in \text{Hol}(\mathbb{D}^d) : \|f\|_{H^2}^2 := \lim_{r \nearrow 1} \int_{\mathbb{T}^d} |f(r\zeta)|^2 dm(\zeta) < \infty \right\}$.
- For ϕ inner, the associated model space $K_\phi := H^2 \ominus \phi H^2$ and the associated compressions of the shift are $S_\phi = P_{K_\phi} M_{z_j}|_{K_\phi}$.

Unitary Characterization (Doubtsov 2020)

The operator $V_\alpha : K_\phi \to L^2(\sigma_\alpha)$ given by

$$V_\alpha \left(\frac{1 - \phi(z)\overline{\phi(w)}}{\prod_{j=1}^d (1 - z_j \bar{w}_j)} \right) = \frac{1 - \alpha\overline{\phi(w)}}{\prod_{j=1}^d (1 - z_j \bar{w}_j)}, \quad \text{for all } w \in \mathbb{D}^d$$

is an isometry. It is unitary iff the polydisk algebra $A(\mathbb{D}^d)$ is dense in $L^2(\sigma_\alpha)$.

Main Question

Finite Blaschke product example: Let $\phi(z) = \prod_{j=1}^{n} \frac{z-a_j}{1-\bar{a}_j z}$, where $a_1, \ldots, a_n \in \mathbb{D}$. Then for $\alpha \in \mathbb{T}$,

$$\sigma_\alpha = \sum_{j=1}^{n} \frac{1}{|\phi'(\zeta_j)|} \delta_{\zeta_j}, \text{ where } \zeta_1, \ldots, \zeta_n \in \mathbb{T} \text{ solve } \phi(\zeta) = \alpha.$$

Question: Can we generalize this to rational inner functions on $\mathbb{D}^2$?

- Paper 1 handles ϕ with $\deg \phi = (m, 1)$. (with J. Cima and A. Sola)
- Paper 2 handles ϕ with $\deg \phi = (m, n)$. (with J. Anderson, L. Bergqvist, J. Cima, A. Sola)

Examples: $\deg \phi = (3, 1)$ and $\deg \psi = (2, 2)$:

$$\phi(z_1, z_2) = \frac{4z_1^3 z_2 - z_1^3 + z_1^2 - 3z_1 - 1}{4 - z_2 + z_1 z_2 - 3z_1^2 z_2 - z_1^3 z_2} \quad \text{and} \quad \psi(z_1, z_2) = \frac{z_1 + z_2 - 3z_2^2 z_1 - 3z_2 z_1^2 + 4z_1^2 z_2^2}{4 - 3z_1 - 3z_2 + z_1^2 z_2 + z_2 z_1^2}$$

Key Background

Recall: Each σ_α is supported on $\mathcal{L}_\alpha =$ the closure of $\{\zeta \in \mathbb{T}^2 : \phi(\zeta) = \alpha\}$.

Key Fact. (B.-Pascoe-Sola, 2018) Let ϕ be rational inner with $\deg \phi = (m, n)$ and let $\alpha \in \mathbb{T}$. Then, there are analytic functions $g_\alpha^1, \dots, g_\alpha^n$ such that $\mathcal{L}_\alpha$ is given by either:

- **Generic Alpha:** $\mathcal{L}_\alpha = \{(\zeta_1, g_\alpha^j(\zeta_1)) : \zeta_1 \in \mathbb{T}, 1 \leq j \leq n\}$.
- **Exceptional Alpha:** $\mathcal{L}_\alpha = \{(\zeta_1, g_\alpha^j(\zeta_1)) : \zeta_1 \in \mathbb{T}, 1 \leq j \leq n\}$ $\cup\{(\tau_k, \zeta_2) : \zeta_2 \in \mathbb{T}, 1 \leq k \leq K\}$ for some $\tau_k \in \mathbb{T}$ and $K \in \mathbb{N}$.

Note: Generically $\phi(\tau, \cdot)$ is a finite Blaschke product with $\deg \phi = n$. If α is exceptional,

- $\phi(\tau_k, \cdot) \equiv \alpha$ is constant for each τ_k.
- ϕ has a singularity on $\mathbb{T}^2$ with z_1-coordinate equal to τ_k for each τ_k.

Example $\mathcal{L}_\alpha$

Let $\phi(z_1, z_2) = \dfrac{4z_1^3 z_2 - z_1^3 + z_1^2 - 3z_1 - 1}{4 - z_2 + z_1 z_2 - 3z_1^2 z_2 - z_1^3 z_2}$ with $\deg \phi = (3, 1)$.

Note: Graph sets in $\mathbb{T}^2$ by identifying each $(\tau_1, \tau_2) \in \mathbb{T}^2$ with $(\mathrm{Arg}(\tau_1), \mathrm{Arg}(\tau_2)) \in (-\pi, \pi]^2$.

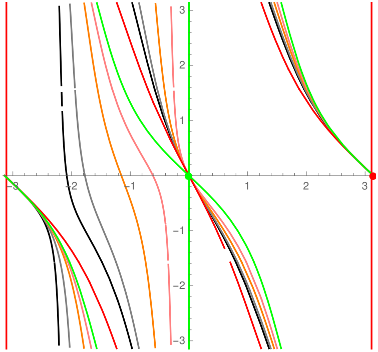

Figure: *The sets $\mathcal{L}_\alpha$ for several generic and exceptional values of α.*

Clark Measure Characterization

Theorem 1 (Anderson-Bergqvist-B.-Cima-Sola, 2023)

Let ϕ be a rational inner function on $\mathbb{D}^2$ and $\alpha \in \mathbb{T}$ with $\deg \phi = (m, n)$. Then σ_α is supported on $\mathcal{L}_\alpha$ and for $f \in C(\mathbb{T}^2)$:

- **Generic Alpha:**

$$\int_{\mathbb{T}^2} f(\zeta) d\sigma_\alpha(\zeta) = \sum_{j=1}^{n} \int_{\mathbb{T}} f\left(\zeta_1, g_\alpha^j(\zeta_1)\right) \frac{1}{\left|\frac{\partial \phi}{\partial z_2}(\zeta_1, g_\alpha^j(\zeta_1))\right|} dm(\zeta_1).$$

- **Exceptional Alpha:**

$$\int_{\mathbb{T}^2} f(\zeta) d\sigma_\alpha(\zeta) = \sum_{j=1}^{n} \int_{\mathbb{T}} f\left(\zeta_1, g_\alpha^j(\zeta_1)\right) \frac{1}{\left|\frac{\partial \phi}{\partial z_2}(\zeta_1, g_\alpha^j(\zeta_1))\right|} dm(\zeta_1)$$

$$+ \sum_{k=1}^{K} \frac{1}{\left|\frac{\partial \phi}{\partial z_1}(\tau_k, z_2)\right|} \int_{\mathbb{T}} f(\tau_k, \zeta_2)\ dm(\zeta_2).$$

Some Notes

- In the degree $(m, 1)$ paper, we worked explicitly with decompositions of $1 - |\phi(z)|^2$ called Agler decompositions and formulas for $\mathcal{L}_\alpha$.
- For α generic, we use the fact that the slices $\phi(\tau, \cdot)$ are finite Blaschke products, various properties of Poisson integrals, and one variable results.
- For α exceptional, we have to carefully decouple the two parts of σ_α.
- One key lemma is showing that $\frac{1}{\left|\frac{\partial \phi}{\partial z_1}(\tau_k, z_2)\right|}$ must be constant.

Higher Dimensions: If ϕ is rational, inner on the polydisk $\mathbb{D}^d$ and continuous on $\overline{\mathbb{D}^d}$, then the two-variable "generic case" arguments generalize and allow us to obtain an analogous characterization of the Clark measures of ϕ.

Unitary Characterization

Recall: The operator $V_\alpha : K_\phi \to L^2(\sigma_\alpha)$ given by

$$V_\alpha \left(\frac{1 - \phi(z)\overline{\phi(w)}}{\prod_{j=1}^d (1 - z_j \bar{w}_j)} \right) = \frac{1 - \alpha\overline{\phi(w)}}{\prod_{j=1}^d (1 - z_j \bar{w}_j)}, \qquad \text{for all } w \in \mathbb{D}^d$$

is an isometry.

Theorem 2 (Anderson-Bergqvist-B.-Cima-Sola, 2023)

Let ϕ be a rational inner function on $\mathbb{D}^2$ and $\alpha \in \mathbb{T}$ with $\deg \phi = (m, n)$. Then V_α is unitary if and only if $\mathcal{L}_\alpha$ contains no vertical or horizontal lines.

Proof Idea: Trig polynomials are dense in $L^2(\sigma_\alpha)$.

- **Most Alpha:** For each trig polynomial p, find $f \in A(\mathbb{D}^2)$ such that $p \equiv f$ on $\mathcal{L}_\alpha$.
- **Alpha with lines:** Show that $\bar{\zeta}_2$ or $\bar{\zeta}_1$ cannot be approximated by $f \in A(\mathbb{D}^2)$.

Connection to Contact Order

The **contact order** of ϕ at a singularity (τ_1, τ_2) on $\mathbb{T}^2$ is an even integer that measures how the zero set of ϕ, $\mathcal{Z}_\phi$, approaches the singularity.

Note: Each branch $\zeta_2 = g_j^\alpha(\zeta_1)$ of $\mathcal{L}_\alpha$ can be associated with a branch of $\mathcal{Z}_\phi$ and that branch has its own contact order K_j.

Theorem 3 (Anderson-Bergqvist-B.-Cima-Sola, 2023)

For all but finitely many $\alpha \in \mathbb{T}$, let $\zeta_2 = g_j^\alpha(\zeta_1)$ be a branch of $\mathcal{L}_\alpha$ going through a singularity (τ_1, τ_2) and let

$$W_j^\alpha(\zeta_1) = \frac{1}{\left|\frac{\partial \phi}{\partial z_2}(\zeta_1, g_\alpha^j(\zeta_1))\right|}.$$

Then there are constants $c_1, c_2 > 0$ with

$$0 < c_1 \leq \frac{W_j^\alpha(\zeta_1)}{|\zeta_1 - \tau_1|^{K_j}} \leq c_2, \quad \text{for } \zeta_1 \text{ near } \tau_1.$$

Example

Let $\phi(z_1, z_2) = \dfrac{4z_1^3 z_2 - z_1^3 + z_1^2 - 3z_1 - 1}{4 - z_2 + z_1 z_2 - 3z_1^2 z_2 - z_1^3 z_2}$ with $\deg \phi = (3, 1)$.

$\phi(-1, \zeta_2) \equiv 1$ and $\phi(1, \zeta_2) \equiv -1$. So $\alpha = 1, -1$ are exceptional.

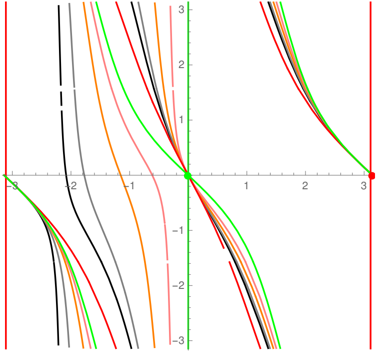

Figure: *Generic level curves $\mathcal{L}_\alpha$ for several values of α (black, gray, orange, pink). Level sets for exceptional values $\alpha = -1$ (green) and $\alpha = 1$ (red).*

Example (cont.)

Generic Alpha: For $\alpha \neq 1, -1$ and $f \in C(\mathbb{T}^2)$,

$$\int_{\mathbb{T}^2} f(\zeta)\sigma_\alpha(\zeta) = \int_{\mathbb{T}} f(\zeta_1, g_\alpha(\zeta_1))W_\alpha(\zeta_1)\ dm(\zeta_1),$$

$$g_\alpha = \frac{1 + 4\alpha + 3\zeta_1 - \zeta_1^2 + \zeta_1^3}{\alpha - \alpha\zeta_1 + 3\alpha\zeta_1^2 + 4z_1^3 + \alpha\zeta_1^3}, W_\alpha = \frac{|\zeta_1 - 1|^2|\zeta_1 + 1|^4}{|4\zeta_1^3 + \alpha\zeta_1^3 + 3\alpha\zeta_1^2 - \alpha\zeta_1 + \alpha|^2}.$$

Exceptional Alpha: For $\alpha = -1$ and $f \in C(\mathbb{T}^2)$, (α=1 is similar)

$$\int_{\mathbb{T}^2} f(\zeta)\sigma_{-1}(\zeta) = \int_{\mathbb{T}} f(\zeta_1, g_{-1}(\zeta_1))W_{-1}(\zeta_1)\ dm(\zeta_1) + \int_{\mathbb{T}} f(1, \zeta_2)\ dm(\zeta_2)$$

$$g_{-1} = \frac{3 + \zeta_1^2}{3\zeta_1^2 + 1}, W_{-1} = \frac{|\zeta_1 + 1|^4}{|3\zeta_1^2 + 1|^2}.$$

The Last Slide

Takeaways

- Clark measures on the polydisk can be defined like those on the disk.
- If ϕ is rational, inner on $\mathbb{D}^2$, then the Clark measures σ_α have a similar structure to those on $\mathbb{D}$ and the associated isometry $V_\alpha : K_\phi \to L^2(\sigma_\alpha)$ is often unitary.
- Other authors have studied other ϕ and other Clark measure generalizations (e.g. Jury 2014, Aleksandrov-Doubtsov 2020, Nell Paiz Jacobsson 2023)

Thanks to the organizers for organizing and the participants for participating!