Weighted norm inequalities for multiplier weak-type inequalities

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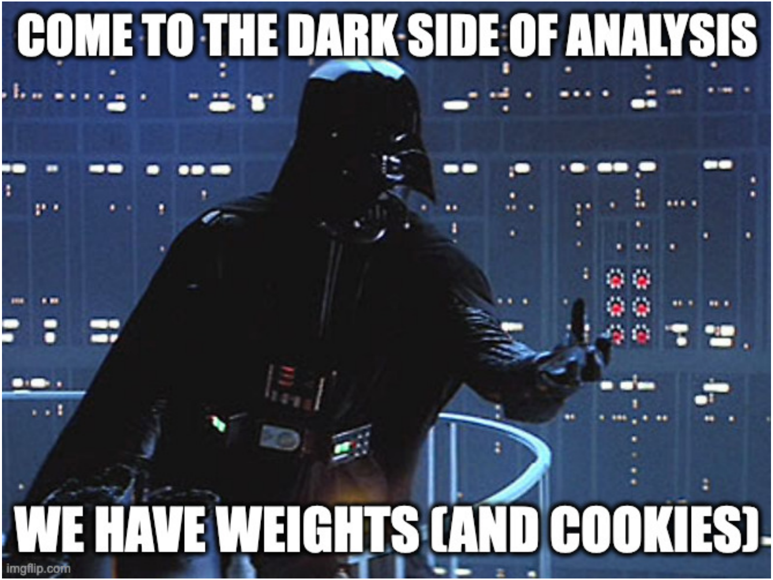
SEAM 40


COME TO THE DARK SIDE OF ANALYSIS
WE HAVE WEIGHTS (AND COOKIES)
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Acknowledgments

Joint work with Brandon Sweeting and Michael Penrod, University of Alabama

Classical weak type inequalities

For $1 \leqslant p < \infty$ we say an operator T satisfies the weak (p,p) inequality if

$$w(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \leqslant \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx.$$

These follow from strong (p,p) inequality by Chebyshev's inequality:

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\,dx \leqslant C \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx.$$

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Muckenhoupt A_p weights

For $1 < p < \infty$, $w \in A_p$ if

$$[w]_{A_p} = \sup_Q \fint_Q w\,dx \left(\fint_Q w^{1-p'}\,dx \right)^{p-1} < \infty.$$

When $p = 1$, $w \in A_1$ if

$$[w]_{A_1} = \sup_Q \operatorname*{ess\,sup}_{x\in Q} w(x)^{-1} \fint_Q w\,dx < \infty.$$

Classical weighted norm inequalities I

Theorem (The A_2 conjecture)

For $1 < p < \infty$ and $w \in A_p$, if T is an SIO,

$$\left(\int_{\mathbb{R}^n} |Tf|^p w\,dx\right)^{\frac{1}{p}} \leqslant C[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}} \left(\int_{\mathbb{R}^n} |f|^p w\,dx\right)^{\frac{1}{p}}.$$

The exponent on $[w]_{A_p}$ is sharp.

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True when $p = 1$ with constant $C[w]_{A_1}\log(e + [w]_{A_1})$.

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The exponent on $[w]_{A_p}$ is sharp.

Weights as multipliers

Restate strong (p, p) inequalities: pull weight inside the power and replace f by $w^{-\frac{1}{p}} f$ to get multiplier strong (p, p):

$$\int_{\mathbb{R}^n} |w^{\frac{1}{p}} T(w^{-\frac{1}{p}} f)|^p \, dx \leqslant C \int_{\mathbb{R}^n} |f|^p \, dx.$$

Unweighted inequality for weighted operator $T_w f = w^{\frac{1}{p}} T(w^{-\frac{1}{p}} f)$.

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Multiplier weak-type inequalities

By Chebyshev's inequality, multiplier strong (p, p) implies

$$|\{x \in \mathbb{R}^n : |w^{\frac{1}{p}}(x)T(w^{-\frac{1}{p}}f)(x)| > \lambda\}| \leqslant \frac{C}{\lambda^p}\int_{\mathbb{R}^n} |f(x)|^p\,dx.$$

We refer to this as a **multiplier** weak (p, p) inequality.

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Endpoint multiplier inequality

Theorem (Muckenhoupt-Wheeden 1977, DCU-JMM-CP 2005)

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Same inequality is true if T is replaced by the maximal operator.

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Original motivation

1. A_p condition sufficient but not necessary: if $n = 1$, $p = 1$, this theorem holds if $w(x) = 1/|x|$.
2. Different conditions required for maximal operators and SIOs when $p > 1$.
3. These inequalities arise in interpolation with change of measure (S-W 1958)

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Quantitative estimates

Question: what is the sharp dependence on $[w]_{A_p}$?

New motivation

Multiplier weak type inequalities allow us to define weak type inequalities for matrix weights.

Matrix weights

Let $\mathcal{S}_d$ be $d \times d$, self-adjoint, positive semi-definite matrices.

A matrix weight is a measurable function

$$W : \mathbb{R}^n \to \mathcal{S}_d.$$

$$|W(x)|_{\mathrm{op}} = \sup_{|\xi|=1} |W(x)\xi|.$$

$$\|\mathbf{f}\|_{L^p(W)} = \left(\int_{\mathbb{R}^n} |W^{\frac{1}{p}}(x)\mathbf{f}(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

Matrix A_p

For $1 < p < \infty$, $W \in \mathbf{A}_p$ if

$$[W]_{\mathbf{A}_p} = \sup_Q \fint_Q \left(\fint_Q |W(x)^{\frac{1}{p}} W(y)^{-\frac{1}{p}}|_{\text{op}}^{p'} \, dy \right)^{\frac{p}{p'}} dx < \infty.$$

$W \in \mathbf{A}_1$ if

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Matrix weights and SIOs

Theorem (NTV, CG, NPTV, DCU-JI-KM)

If $1 < p < \infty$, $W \in \mathbf{A}_p$, and T an SIO, then

$$\left(\int_{\mathbb{R}^n} |W^{\frac{1}{p}}(x) T(W^{-\frac{1}{p}}\mathbf{f})(x)|^p \, dx \right)^{\frac{1}{p}} \leqslant C[W]_{\mathbf{A}_p}^{1+\frac{1}{p-1}-\frac{1}{p}} \left(\int_{\mathbb{R}^n} |\mathbf{f}(x)|^p \, dx \right)^{\frac{1}{p}}.$$

This exponent is sharp when $p = 2$ (DPTV 2024).

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Matrix weights and Maximal operator

Christ-Goldberg maximal operator

$$M_W f(x) = \sup_Q \fint_Q |W^{\frac{1}{p}}(x) W^{-\frac{1}{p}}(y)\mathbf{f}(y)|\,dy \cdot \chi_Q(x).$$

Theorem (MC-MG 2003, KM-JI 2019)

If $1 < p < \infty$, $W \in \mathbf{A}_p$,

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Theorem (DCU-JI-KM-SP-IRR 2021)

For $1 \leqslant p < \infty$ and $w \in A_1$, if T is an SIO,

$$|\{x \in \mathbb{R}^n : |W(x)T(W^{-1}\mathbf{f})(x)| > \lambda\}| \leqslant \frac{C[W]_{A_1}^2}{\lambda} \int_{\mathbb{R}^n} |\mathbf{f}(x)|\,dx.$$

The same inequality holds for M_W.

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Short outline of proof

1. Use convex-body sparse domination (NPTV 2017)
2. Use alternative formulation of $L^{p,\infty}$ norm

$$\|f\|_{L^{p,\infty}} \approx \sup_{\substack{E\subset\mathbb{R}^n \\ 0<|E|<\infty}} \inf_{\substack{F\subset E \\ |F|\geqslant\frac{1}{2}|E|}} |E|^{-1+\frac{1}{p}} \left| \int_F f(x)\,dx \right|.$$

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Quantitative estimates II, $1 < p < 2$

Theorem (AL-KL-SO-IRR 2024)

For $1 \leqslant p < 2$ and $W \in A_p$

$$\|M_W \mathbf{f}\|_{L^{p,\infty}} \leqslant [W]_{A_p}^{\frac{2}{p}} \|\mathbf{f}\|_{L^p}$$

and the exponent is sharp.

In the scalar case, if T is an SIO

$$\|w^{\frac{1}{p}} T(w^{-\frac{1}{p}} f)\|_{L^{p,\infty}} \leqslant [w]_{A_p}^{1+\frac{1}{p^2}} \log(e + [w]_{A_p})^{\frac{1}{p}} \|f\|_{L^p}$$

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Quantitative estimates II, $p \geqslant 2$

In addition, they proved when $p \geqslant 2$:

1. Best exponent in scalar case for Hilbert transform is 1, as gotten from strong (p, p) inequality and Chebyshev's inequality.

2. Best constant for maximal operator is bounded below by

$$[w]_{A_p}^{\frac{1}{p-1}} \log(e + [W]_{A_p})^{-\frac{1}{p}}.$$

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Two operators

Fractional integral operator: $0 < \alpha < n$

$$I_\alpha \mathbf{f}(x) = \int_{\mathbb{R}^n} \frac{\mathbf{f}(y)}{|x-y|^{n-\alpha}}\,dy$$

Fractional Christ-Goldberg maximal operator:

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Matrix $A_{p,q}$ weights

$W \in A_{p,q}$, $1 < p < \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ if

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Strong (p, q) inequalities

Theorem (JI-KM 2019)

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and this exponent is sharp.

$$\|WI_\alpha(W^{-1}\mathbf{f})\|_{L^q} \leqslant C[W]_{A_{p,q}}^{(1-\frac{\alpha}{n})\frac{p'}{q})+1} \|\mathbf{f}\|_{L^p}.$$

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Sharp strong (p, q) constants

Open question: find the sharp exponent for I_α in matrix case.

Known in scalar case:

$$\left(1 - \frac{\alpha}{n}\right) \max\left(1, \frac{p'}{q}\right).$$

(ML-KM-CP-RT 2010)

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Multiplier weak (p, q) inequalities

Theorem (DCU-BS 2024)

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The same inequality holds for $M_{W,\alpha}$.

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Open question: find the sharp exponent for I_α and $M_{W,\alpha}$ in matrix case.

A better definition of matrix weights

For $1 < p < \infty$, $W \in \widehat{A}_p$ if

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$W \in \widehat{A}_p$ if and only if $W^p \in A_p$, and

$$[W]_{\widehat{A}_p} = [W^p]_{A_p}^{\frac{1}{p}}.$$

Left/right openness of $\widehat{A}_p$

Theorem (DCU-MP 2023)

If $1 < p < \infty$ and $W \in \widehat{A}_p$, there exists $\delta = \delta([W]_{\mathbf{A}_p}) > 0$ such that $|q - p| < \delta$, then $W \in \widehat{A}_q$.

Contrast this with

Theorem (MB 2001)

Given $1 < p < \infty$, there exists $W \in A_p$ such that W is not in A_q, any $q < p$.

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Bounds for SIOs

Theorem (DCU-MP-BS 2024)

If $1 < p < \infty$, $W \in \widehat{A}_p$, and T an SIO, then

$$\left(\int_{\mathbb{R}^n} |W(x) T(W^{-1}\mathbf{f})(x)|^p \, dx \right)^{\frac{1}{p}} \leqslant C \left(\int_{\mathbb{R}^n} |\mathbf{f}(x)|^p \, dx \right)^{\frac{1}{p}}.$$

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Same result holds for M_W.

Sketch of proof

1. $T_W\mathbf{f} = WT(W^{-1}\mathbf{f})$ is a linear operator independent of p
2. $W \in \widehat{A}_p$ implies $W \in \widehat{A}_{p\pm\epsilon}$ for some $\epsilon > 0$.
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Thank You!

Roll Tide!