On a Problem of von Renteln

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Problem 5.62. Let $D = \{z : |z| < 1\}$, $T = \{z : |z| = 1\}$ and u be a continuous real-valued function on T. Give a necessary and sufficient condition on u such that u is the real part of a function f in the disc algebra $A(\overline{D})$.

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Remarks.
(1) A solution would have applications in the algebraic ideal theory.
(2) An answer to the analogous problem for $L^p(T)$, $H^p(D)$ is the Burkholder-Gundy-Silverstein Theorem (see Peterson [3], p. 13).

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It has been unknown for years but a solution to Problem 5.62 is formulated in A. Zygmund's book [5], however, without a proof. For the proof, the book [5] suggests a hint, but the hint seems to be irrelevant, unfortunately.

More precisely, the proposition 5, part (a), p. 180, in [5] formulates a solution to Problem 5.62 with a reference to a report of M. Zamansky [4] which contains no proofs. Perhaps Zamansky never published his proof (otherwise Zygmund would give a reference to it).

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Indeed, Zygmund [5] (p.180) gives a parenthetical hint for a proof by suggesting to use uniformity in the relation (3.21) of Chapter III of [5]. But (3.21) does not contain a uniformity case and it is not clear how to give a proof following the hint.

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As in [5], we use the standard notations $\tilde{f}(r, x)$ and $\tilde{f}(x)$ for the conjugate functions in the unit disc and on the (interval $[0, 2\pi]$ of) real line, respectively.

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As in [5], we use the standard notations $\tilde{f}(r, x)$ and $\tilde{f}(x)$ for the conjugate functions in the unit disc and on the (interval $[0, 2\pi]$ of) real line, respectively.

Also, following [5], we use the notation

$$\tilde{f}(x; h) = -\frac{1}{\pi} \int_h^{\pi} [f(x+t) - f(x-t)] \frac{1}{2} \cot \frac{1}{2} t \; dt.$$

We need the following classical theorem on the boundary behavior of the conjugate harmonic function (see Theorem (7.20) in [5], p. 103).

Theorem A. *If f is integrable and F the indefinite integral of f, then*

$$\tilde{f}(r,x) - \left(-\frac{1}{\pi}\int_{1-r}^{\pi}[f(x+t)-f(x-t)]\frac{1}{2}\cot\frac{1}{2}t\ dt\right) \to 0 \quad (r \to 1)$$

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By the definition (see the top of the same p. 103 in [5]), the function F is smooth at the point x if $F(x+t)+F(x-t)-2F(x)=o(t)$. Since obviously the existence of the finite derivative of F at x implies the smoothness of F at x, the limit relation in Theorem A is true a. e.

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Zamansky's Theorem. *Let $f(x)$ be continuous and periodic. A necessary and sufficient condition for $\tilde{f}(x)$ to be continuous is that*

$$\tilde{f}(x;h) = -\frac{1}{\pi}\int_h^\pi [f(x+t) - f(x-t)]\frac{1}{2}\cot\frac{1}{2}t\ dt$$

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Before proving Zamansky's theorem, note that in the formulation of this theorem we presented $\tilde{f}(x;h)$ in slightly more explicit form than in [5], p. 180. To see that the above form is the same as that in [5], we refer the reader to the simple notation adopted on p. 50 in [5].

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exists for almost all x (see Theorem (3.1), p. 131, [5]).

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exists for almost all x (see Theorem (3.1), p. 131, [5]).
By the definition of $\tilde{f}(x;h)$, the previous equality can be written as $\tilde{f}(x) = \lim_{h\to 0}\tilde{f}(x;h)$ for almost all x. Since $\tilde{f}(x;h)$ converges uniformly, $\tilde{f}(x)$ is continuous.

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We write the left part of the limit relation in Theorem A as follows:

$$[\tilde{f}(r,x)-\tilde{f}(x)]+\left[\tilde{f}(x)-\left(-\frac{1}{\pi}\int_{1-r}^{\pi}[f(x+t)-f(x-t)]\frac{1}{2}\cot\frac{1}{2}t\ dt\right)\right]\rightarrow$$

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Since f is continuous, by Theorem A the convergence in the last limit relation is uniform.
Since $\tilde{f}(x)$ is continuous, we have that $[\tilde{f}(r,x)-\tilde{f}(x)]\to 0 \quad (r\to 1)$ uniformly. Thus

$$\left[\tilde{f}(x)-\left(-\frac{1}{\pi}\int_{1-r}^{\pi}[f(x+t)-f(x-t)]\frac{1}{2}\cot\frac{1}{2}t\ dt\right)\right]\to 0\ \ (r\to 1)$$

uniformly too.

Because $\tilde{f}(x)$ does not depend on r, the convergence of the term

$$\left(-\frac{1}{\pi}\int_{1-r}^{\pi}[f(x+t)-f(x-t)]\frac{1}{2}\cot\frac{1}{2}t\,dt\right)\quad(r\to 1)$$

is uniform, which is the same as
$\tilde{f}(x;h)=-\frac{1}{\pi}\int_{h}^{\pi}[f(x+t)-f(x-t)]\frac{1}{2}\cot\frac{1}{2}t\,dt$ converges uniformly as $h\to+0$. This completes the proof.

D. M. Campbell, J. G. Clunie, and W. K. Hayman, Research problems in complex analysis, In: Aspects of contemporary complex analysis, Proc. instr. Conf., Durham/Engl. 1979, 527-572 (1980).

W. K. Hayman and E. F. Lingham, Research Problems in Function Theory, Fiftieth Anniversary Edition, Springer, 2019.

K. E. Peterson, Hardy spaces and bounded mean oscillation, Cambridge University Press, Cambridge, 1977.

M. Zamansky, Sur l'approximation des functions continues, Comptes Rendus Acad. Sci., **228** (1949), 460-461.

A. Zygmund, Trigonometric series, V. 1, Cambridge, 1959.

Thank You!