Projections in the combination of Operators of Finite Orders

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Outline

1. Introduction
2. Preliminaries
3. Projections as averages of isometries & reflections
4. Ongoing and Future Plans

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- Isometries and projections are interconnected.

Definitions & Examples

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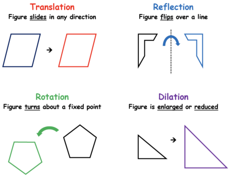

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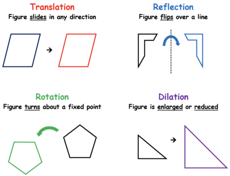

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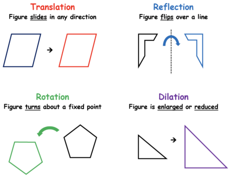

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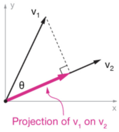

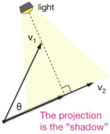

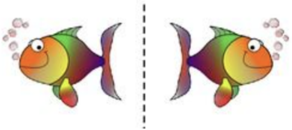

Creating Projections from reflections (and isometries)

Example

Consider $\mathbb{R}^3$ and consider the norm

$$\|(x_1, x_2, x_3)\|_\infty = \max\{|x_1|, |x_2|, |x_3|\}.$$

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Note: P is a projection!

An example in the infinite dimensional world!

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Consider the space $C[0,1]$, the space of all continuous functions on the interval $[0,1]$ with the norm $\|f\|_\infty = \max_{t\in[0,1]} f(t)$.

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Remark

In general, if R is an (isometry), and a reflection, then the operator P defined as the average of I and R (i.e., $P = \frac{1}{2}(I + R)$) is a projection.

Projections as a combination of two isometries

(DBE, 2023) For two isometries T_0 and T_1 on a Banach space X, consider an operator Q defined as $Q = \lambda_0 T_0 + \lambda_1 T_1$ $(Q \neq I)$ with $\lambda_0, \lambda_1 > 0$ and $\lambda_0 + \lambda_1 = 1$. If Q is a projection, then $\lambda_0 = \lambda_1 = \frac{1}{2}$.

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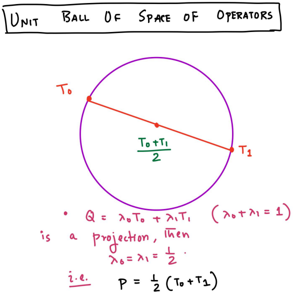

Projections in the “convex hull” of operators of finite order

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$$Q = \lambda_0 I + \lambda_1 T + \lambda_2 T^2 + \cdots + \lambda_{n-1} T^{n-1}.$$

Then Q is a projection if and ony if $\lambda_0 = \lambda_1 = \cdots = \lambda_{n-1} = \dfrac{1}{n}$.

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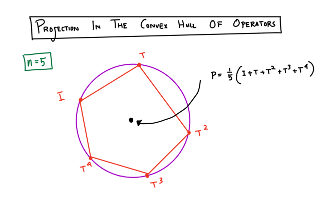

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For n=2 and beyond

For $n = 2$, let T be such that $T^2 = T$. Then

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T	"eigen-projection"
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T^2	$*$

Observation: $(*)$ is the sum of the eigen-projections!

Eigen-projections and Examples

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Definition (Eigen-projection)

Let T be a bounded operator on X and let λ be an eigenvalue of T. Let U be an open set in $\mathbb{C}$ such that $\sigma(T) \subset U$. Let $\Gamma_\lambda :\to U \setminus \sigma(T)$ be a loop that contains λ in its interior.

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$$P_\lambda := -\frac{1}{2\pi i} \int_{\Gamma_\lambda} (T - zI)^{-1} \, dz \tag{1}$$

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Example

For $n = 3$, let T be such that $T^3 = T$.

Eigenvalue	Eigen-projection
0	$I - T^2$
1	$\frac{1}{2}(T + T^2)$
-1	$\frac{1}{2}(-T + T^2)$

List of projections for n=4 and the general case

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Example

For $n = 4$, let T be such that $T^4 = T$.

Number	$n = 3$	Comments
1	$0, I$	"trivial projection"
2	$\frac{1}{3}(T + T^2 + T^3)$	"eigen-projection"
3	$\frac{1}{3}(\omega T + \omega^2 T^2 + T^3)$	"eigen-projection"
4	$I - T^3$	"eigen-projection"
5	$\frac{1}{3}(\omega^2 T + \omega T^2 + T^3)$	"eigen-projection"
6	$(\alpha T + \beta T^2 + \frac{2}{3} T^3)$	"(#2)+(#3)"
7	$(\beta T + \alpha T^2 + \frac{2}{3} T^3)$	"(#2)+(#5)"
8	$\frac{1}{3}(-T - T^2 + 2T^3)$	"(#3)+(#5)"
9	T^3	"(#2)+(#3)+(#5)"

where, $\alpha = \frac{1+\sqrt{3}i}{6}$, $\beta = \frac{1-\sqrt{3}i}{6}$, $\omega = \frac{-1+\sqrt{3}i}{2}$.

The General Case

For an n-potent operator T (i.e., $T^n = T$), is it possible to classify all the projections in the combination of operators $\{I, T, T^2, \cdots, T^{n-1}\}$ in terms of the "eigen-projections?"

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(DBE 2024) For a projection P in the combination of powers of T (i.e., $P = a_0 I + a_1 T + a_2 T^2 + \cdots + a_{k-1} T^{k-1}$) we have

$$P = \sum_{j:\omega^j \in \sigma(T)} \lambda_j P_j,$$

where λ_j are scalars of modulus 1 and P_j is an eigen-projection associated with the eigenvalue ω^j.


thank you!