Fejér–Riesz Factorization
Noncommutative and Multivariable Perspectives

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Noncommutative Fejér–Riesz Factorization
NC multivariate perspectives

- Introduction.
  - Orientation: operator-valued Fejér–Riesz ($\mathbb{Z}$).
  - Dritschel's multivariate version ($\mathbb{Z} \times \mathbb{Z}^g$) – motivating template.
  - Mixed variable FR formulated.
- Fejér-Riesz for products $\mathscr{W} \times G$.
  - $\mathscr{W}$ is the free semigroup or $\mathbb{Z}_2^{*g}$;
  - A nearly perfect special case.
- Ingredients.
  - The Parrott Lemma for functions on G.
  - The $\mathbb{Z}_2^{*g}$ completion problem.
- Concluding remarks: bonus content.

The Fejér–Riesz Theorem
Rosenblum's version

Given a positive integer d and operators

$$p_{-d}, \ldots, p_{-1}, p_0, p_1, \ldots, p_d \in B(\mathscr{H}),$$

on a Hilbert space $\mathscr{H}$, if the *trigonmetric polynomial*

$$p(t) = \sum_{j=-d}^{d} p_j e^{ijt} \succeq 0$$

is psd for all real t, then there exists an *analytic polynomial*

$$q(t) = \sum_{j=0}^{d} q_0 e^{ijt}, \quad q_j \in B(\mathscr{H}),$$

such that

$$p(t) = q(t)^* q(t), \quad p = q^* q.$$

The Fejér–Riesz Theorem
Rosenblum's version

If

$$p(t) = \sum_{j=-d}^{d} p_j e^{ijt} \succeq 0, \quad p_n \in B(\mathscr{H}),$$

then $p = q^* q$, for some

$$q(t) = \sum_{j=0}^{d} e^{ijt}, \quad q_j \in B(\mathscr{H}).$$

(i) Elementary in the scalar case of $\mathscr{H} = \mathbb{C}$.

(ii) The factorization is *perfect* - nonnegative suffices, optimal degree, $B(\mathscr{H})$-coefficients.

(iii) The converse is obvious - an algebraic certificate of positivity.

(iv) Can interpret as a sums of squares/posstatz result.

(v) The underlying group is $\mathbb{Z}$ with irr reps $e^{it} \in \mathbb{T}$.

The Fejér–Riesz Theorem
Dritschel's several variable version

For $t = (t_0, (t_1, \dots, t_g)) \in \mathbb{R} \times \mathbb{R}^g$ and $n = (n_0, (n_1, \dots, n_g)) \in \mathbb{Z} \times \mathbb{Z}^g$,

$$\langle n, t \rangle = t_0 n_0 + \sum_{j=1}^{g} t_j n_j.$$

Theorem. [Dritschel, 2004] If the *multivariate* trig poly

$$p(t) = \sum_{n=-d}^{d} p_n e^{i\langle n,t \rangle} \succeq \epsilon > 0, \quad p_n \in B(\mathcal{H})$$

is strictly positive, then

$$p(t) = q(t)^* q(t),$$

for some analytic polynomial,

$$q(t) = \sum_{n=0}^{N} q_n e^{i\langle n,t \rangle}, \quad q_n \in \mathcal{B}(\mathcal{H}, \mathcal{F}).$$

Dritschel's several variable FR theorem

Finer detail

For $d = (d_0, (d_1, \dots, d_g)) \in \mathbb{N} \times \mathbb{N}^g$, if

$$p(t) = \sum_{n=-d}^{d} p_n e^{i\langle n,t\rangle} \succeq \epsilon > 0, \quad p_n \in B(\mathcal{H})$$

is strictly positive, then for for some $M \in \mathbb{N}^g$ and $N = (d_0, M)$,

$$p(t) = q(t)^* q(t), \quad q(t) = \sum_{n=0}^{N} q_n e^{i\langle n,t\rangle}, \quad q_n \in \mathcal{B}(\mathcal{H}, \mathcal{F}).$$

(i) The group here is $\mathbb{Z} \times \mathbb{Z}^g$ its irr reps (parameterized by) $\mathbb{T} \times \mathbb{T}^g$.

(ii) *Partially perfect*: q has degree d_0 in the t_0 variable.

(iii) Relatively concrete degree bounds in terms of the data.

(iv) The dimension of $\mathcal{F}$ can exceed that of $\mathcal{H}$ (a true SoS).

Mixed variable FR Factorization

Dritschel's multivariate FR factorization suggests the paradigm:

`Magical Thinking Meta Theorem.` Given a group $\mathscr{W}$ that supports a perfect FR theorem and a group G, the group $\mathscr{W} \times G$ supports a partially perfect FR theorem.

- Reality check: For $\mathbb{Z} \times \mathbb{Z}$ something has to go degree, nonnegative, $B(\mathscr{H})$.
- Groups $\mathscr{W}$ that support a perfect FR.
  (i) $\langle x \rangle$, the free (semi)group on variables $x = (x_1, \ldots, x_g)$ - not a group but close enough.

  (ii) $\mathbb{Z}_2^{*g} = \mathbb{Z}_2 * \mathbb{Z}_2 * \cdots * \mathbb{Z}_2$.
- G a discrete group generated by a finite set $y = (y_1, \ldots, y_h)$.
- Positivity (psd): An element $w \in \mathscr{W} \times G$ is *positive* if $\pi(w) \succeq 0$ for all representations of $\mathscr{W} \times G$ on Hilbert space.
- The case $\langle x \rangle \times \mathbb{Z}$ - several unitary NC variables with one commuting variable was our initial motivation.

Mixed variable NC FR

- Let $x = (x_1, \ldots, x_g)$ denote g freely non-commuting variables and let $\langle x \rangle$ denote the free semigroup generated by x;
- Give $\langle x \rangle$ the *shortlex* order - a notion of degree;
- Let $\mathscr{x}_1, \ldots, \mathscr{x}_g$ denote the generators of $\mathbb{Z}_2^{*g}$ so that $\mathscr{x}_j^2 = 1$.
- Give $\mathbb{Z}_2^{*g} = \langle \mathscr{x} \rangle$ the *shortlex* order too.

- Let $\mathscr{W}$ denote either $\langle x \rangle$ or $\mathbb{Z}_2^{*g}$.
- For $\mathscr{d} \in \mathscr{W}$, let $\mathscr{W}_{\mathscr{d}}$ denote the words of length at most $\mathscr{d}$;
- Left fractions:

$$\ell\text{-Frac}\,\mathscr{W} = \{u^{-1}v : u, v \in \mathscr{W}\}, \quad \ell\text{-Frac}\,\mathscr{W}_{\mathscr{d}} = \{u^{-1}v : u, v \in \mathscr{W}_{\mathscr{d}}\}.$$

- In either case, the unitary reps of $\mathscr{W}$, and hence of $\mathscr{W} \times G$, are easily described: $x_j \mapsto U_j$ or $\mathscr{x}_j \mapsto U_j$ with $U_j^2 = I$.

Mixed variable NC FR

- A *trig polynomial* of degree (at most) $\mathscr{d}$ takes the form,

$$p = \sum_{w \in \ell\text{-Frac}\,\mathscr{W}_{\mathscr{d}} \times G}^{\text{finite}} p_w w, \quad p_w \in M_K(\mathbb{C}).$$

- An *analytic polynomial* of degree (at most) $\mathscr{d}$ has the form,

$$q = \sum_{u \in \mathscr{W}_{\mathscr{d}} \times \langle y \rangle}^{\text{finite}} q_u u, \quad q_u \in M_{K',K}(\mathbb{C}).$$

Theorem. [KLM] If p has degree $\mathscr{d}$ and is strictly psd,

$$p(\pi) = \sum_{w \in \ell\text{-Frac}\,\mathscr{W}_{\mathscr{d}} \times G}^{\text{finite}} p_w \otimes \pi(w) \succeq \epsilon > 0, \quad \text{for all } \pi,$$

then there exists an analytic q of of degree $\mathscr{d}$ such that $p = q^* q$.

Mixed variable NC FR

Theorem. [KLM] If p has degree d and is strictly psd,

$$p(\pi) = \sum_{w \in \ell\text{-Frac}\,\mathscr{W}_d \times G}^{\text{finite}} p_w \otimes \pi(w) \succeq \epsilon > 0,$$

then there exists

$$q = \sum_{u \in \mathscr{W}_d \times \langle y \rangle}^{\text{finite}} q_u u, \quad q_u \in M_{K',K}(\mathbb{C}).$$

such that $p = q^* q$.

- $\langle y \rangle = \mathbb{N}^{\mathrm{g}}$ when $G = \mathbb{Z}^{\mathrm{g}}$. gives mat-valued case of Dritschel's result.

Proof outline:

- Parrott's lemma for matrices whose entries are functions on G;
- Solving an op-sys matrix completion problem gives a cp map;
- Arveson/Stinspring and Choi matrix magic produces q.

Mixed variable NC FR for finite G is nearly perfect

Theorem. [KLM] For $\mathscr{W}$ either $\langle x \rangle$ or $\mathbb{Z}_2^{*g}$, if G is finite, p has degree $\mathscr{d}$ and

$$p(\pi) = \sum_{w \in \ell\text{-Frac}\,\mathscr{W}_{\mathscr{d}} \times G}^{\text{finite}} p_w \otimes \pi(w) \succeq 0, \quad p_w \in B(\mathscr{H}),$$

for all representations π, then there exists

$$q = \sum_{u \in \mathscr{W}_{\mathscr{d}} \times G}^{\text{finite}} q_u u, \quad q_u \in M_{K',K}(\mathbb{C}).$$

such that $p = q^* q$.

- The case $\mathscr{W} = \langle x \rangle$ and G trivial is well known with many trills.

Parrott for functions on a group

A function $\tau : G \to M_K(\mathbb{C})$ determines a *Toeplitz* matrix,

$$\Upsilon_\tau = \big(\tau(g^{-1}h)\big)_{g,h\in G}$$

and we view this matrix as a form on $G \times G$ defined by

$$\langle \Upsilon_\tau \psi, \phi\rangle = \sum_{g,h\in G} \langle \tau(g^{-1}h)\psi(h), \phi(g)\rangle,$$

for $\psi, \phi \in C_{0,0}(G, \mathbb{C}^K)$, the $\mathbb{C}^K$-valued functions of finite support.

- In the case $G = \mathbb{Z}$, where the group operation is addition,

  $$\Upsilon_\tau = \big(\tau(j-i)\big)_{i,j\in\mathbb{Z}}$$

  is an actual Teoplitz matrix.

Parrott for functions on a group

Proposition. Given $\tau_{i,j} : G \to M_k(\mathbb{C})$, if

$$P = \begin{pmatrix} \tau_{0,0} & \tau_{0,1} & \cdots & \tau_{0,N-1} \\ \tau_{1,0} & \tau_{1,1} & \cdots & \tau_{1,N-1} \\ \vdots & \vdots & \cdots & \vdots \\ \tau_{N-1,0} & \tau_{N-1,1} & \cdots & \tau_{N-1,N-1} \end{pmatrix} \succeq 0$$

and

$$Q = \begin{pmatrix} \tau_{1,1} & \cdots & \tau_{1,N-1} & \tau_{1,N} \\ \vdots & \cdots & \vdots & \vdots \\ \tau_{N-1,1} & \cdots & \tau_{N-1,N-1} & \tau_{N-1,N} \\ \tau_{N,1} & \cdots & \tau_{N,N-1} & \tau_{N,N} \end{pmatrix} \succeq 0,$$

then there is a function $\tau_{0,N} : G \to M_k(\mathbb{C})$ such that

$$R = \begin{pmatrix} \tau_{0,0} & \tau_{0,1} & \cdots & \tau_{0,N-1} & \tau_{0,N} \\ \tau_{0,1} & \tau_{1,1} & \cdots & \tau_{1,N-1} & \tau_{1,N} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \tau_{N-1,0} & \tau_{N-1,1} & \cdots & \tau_{N-1,N-1} & \tau_{N-1,N} \\ \tau_{0,N}^* & \tau_{1,N} & \cdots & \tau_{N-1,N} & \tau_{N,N} \end{pmatrix} \succeq 0.$$

Parrott for functions on a group ...
... as used

- Let $F(G)$ denote the functions $\tau : G \to M_K(\mathbb{C})$;
- Given A, B, C, D, E with entries from $F(G)$.

If

$$P = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \succeq 0, \quad Q = \begin{pmatrix} C & D \\ D^* & E \end{pmatrix} \succeq 0,$$

then there exists X with entries from $F(G)$ such that

$$\begin{pmatrix} A & B & X \\ B^* & C & D \\ X^* & D^* & E \end{pmatrix} \succeq 0$$

- In interpolation/factorization: identify Q as, up to unitary equivalence (a permutation), a submatrix of P. So $P \succeq 0$ suffices.

Matrix completions for $\mathbb{Z}_2^{*g}$

- Let $F(G)$ denote the set of function from G to $M_K(\mathbb{C})$.
- Let $\mathscr{d} \in \mathscr{W}$ be given and let $\mathscr{s}$ denote its immediate successor.
- $\mathscr{s} = x_j \mathscr{z}$.
- Suppose $\chi : \ell\text{-Frac}\,\mathscr{W}_{\mathscr{d}} \to F(G)$ is a partially defined function.

Prop'n. [KJM] If the *partial toeplitz* matrix with toeplitz entries,

$$\left(\chi(u^{-1}v)\right)_{u,v\in\mathscr{W}_{\mathscr{d}}} \succeq 0,$$

is psd, then χ extend to a map $\overline{\chi} : \ell\text{-Frac}\,\mathscr{W}_{\mathscr{s}} \to F(G)$ such that

$$\left(\overline{\chi}(u^{-1}v)\right)_{u,v\in\mathscr{W}_{\mathscr{s}}} \succeq 0.$$

Matrix completions for $\mathbb{Z}_2^{*g}$

Prop'n. If $\mathscr{s}$ is the successor to $\mathscr{d}$ and the *partial toeplitz* matrix with toeplitz entries,

$$\left(\chi(u^{-1}v)\right)_{u,v\in\mathscr{W}_{\mathscr{d}}} \succeq 0,$$

is psd, then χ extend to a mapping $\overline{\chi} : \ell\text{-Frac}\,\mathscr{W}_{\mathscr{s}} \to F(G)$ such that

$$\left(\overline{\chi}(u^{-1}v)\right)_{u,v\in\mathscr{W}_{\mathscr{s}}} \succeq 0.$$

- $\mathscr{s} = x_j\mathscr{z}$ successor to $\mathscr{d}$;
- $L = \mathscr{W}_{\mathscr{z}} \cup x_j\mathscr{W}_{\mathscr{z}}$. Note $L \subseteq \mathscr{W}_{\mathscr{s}}$ and $L \setminus \{\mathscr{s}\} \subseteq \mathscr{W}_{\mathscr{d}}$.
- The map $m_{x_j} : L \to L$ defined by $m_{x_j}w = x_jw$ is a bijection that interchanges $\mathscr{z}$ and $\mathscr{s}$: $\mathbb{Z}_2$ is special.
- Validates Parrott with one caveat: the $(\mathscr{z},\mathscr{s})$ and $(\mathscr{s},\mathscr{z})$ entries correspond to $\mathscr{z}x_j\mathscr{z}$ must agree.

Bonus content
Play the CHSH quantum game

- The strict generalization from $\mathbb{Z}_2^{*g}$ to, say, $\mathbb{Z}_2 * \mathbb{Z}_3$ fails.
- A reason to care: Violation of a Bell inequality signals entanglement and implies that a physical interaction cannot be explained by any classical picture of physics:
- The Clauser-Horne-Shimony-Holt (CHSH) inequality, $(\mathbb{Z}_2 * \mathbb{Z}_2) \times (\mathbb{Z}_2 * \mathbb{Z}_2)$. Letting x_1, x_2 and y_1, y_2 denote the generators, $x_j \leftrightarrow y_k$,

$$\text{CHSH}: \quad x_1 y_1 + x_1 y_2 + x_2 y_1 - x_2 y_2 \leq 2\sqrt{2}.$$

- The optimal value is 2 when $x_j, y_j \in \mathbb{R}$.
- The bound is attained with the 4×4 matrices,

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I_2, \; X_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes I_2, \; Y_1 = I_2 \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \; Y_2 = I_2 \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

Bonus content

More proof - sorry

- Strict positivity, absence of degree bounds, and mat-valued: A nested sequence compactness argument shows there exists a $d \in \langle y \rangle$ such that, letting S denote the operator system

$$\left(X_{u^{-1}v,g^{-1}h}\right)_{u,v \leq d,\, g,h \leq d},$$

the map $\Phi_p : S \to M_K(\mathbb{C})$ defined by

$$\Phi_p(X) = \sum_{(u,h) \in \ell\text{-Frac}\,\mathscr{W}_d \times \ell\text{-Frac}\langle y \rangle_d} X_{u,h}\, p_{u,h}$$

is completely positive.

- Assuming no such g exists leads to a partially defined psd function $\overline{X} \sim \psi : \ell\text{-Frac}\,\mathscr{W}_d \times G \to M_K(\mathbb{C})$ and then, by solving the matrix completion problem, a representation π giving the contradiction $p(\pi) \not\succeq 0$.