Swap Operators and the Quantum Max Cut Problem

Igor Klep University of Ljubljana, Slovenia

SEAM 40
The 40th South Eastern Analysis Meeting, Gainesville, FL
March 2024

Based on joint works with

Adam Bene Watts	U Waterloo
Anirban Chowdhury	U Waterloo
Aidan Epperly	UC Davis
Bill Helton	UC San Diego
Tea Štrekelj	U Ljubljana
Jurij Volčič	Drexel U

Outline

Quantum Max Cut
- Pauli matrices
- 2-local Hamiltonian problem
- Physics motivation

Swap operators
- Schur-Weyl duality
- Helton-McCullough Positivstellensatz
- Numerical examples
- Solving a moment problem approximately

Exact solutions
- Clique
- Star graph
- An algorithm

Takeaway messages

IK's work was performed within the project COMPUTE, funded within the QuantERA II Programme that has received funding from the EU's H2020 research and innovation programme under the GA No 101017733

Quantum Max Cut

Pauli matrices

The Pauli matrices are the following three self-adjoint 2×2 matrices

$$\sigma_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{Pauli}$$

Quantum Max Cut

Pauli matrices

The Pauli matrices are the following three self-adjoint 2×2 matrices

$$\sigma_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{Pauli}$$

Their multiplication table is as follows:

	σ_X	σ_Y	σ_Z
σ_X	I_2	$i\,\sigma_Z$	$-i\,\sigma_Y$
σ_Y	$-i\,\sigma_Z$	I_2	$i\,\sigma_X$
σ_Z	$i\,\sigma_Y$	$-i\,\sigma_X$	I_2

Quantum Max Cut

Pauli matrices

The Pauli matrices are the following three self-adjoint 2×2 matrices

$$\sigma_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \qquad \text{(Pauli)}$$

For $W \in \{X, Y, Z\}$ and $k, n \in \mathbb{N}$ we shall also use

$$\sigma_W^k = \underbrace{I_2 \otimes \cdots \otimes I_2}_{k-1} \otimes \sigma_W \otimes \underbrace{I_2 \otimes \cdots \otimes I_2}_{n-k} \in M_2(\mathbb{C})^{\otimes n} = M_{2^n}(\mathbb{C}).$$

Quantum Max Cut

Pauli matrices

The Pauli matrices are the following three self-adjoint 2×2 matrices

$$\sigma_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{Pauli}$$

For $W \in \{X, Y, Z\}$ and $k, n \in \mathbb{N}$ we shall also use

$$\sigma_W^k = \underbrace{I_2 \otimes \cdots \otimes I_2}_{k-1} \otimes \sigma_W \otimes \underbrace{I_2 \otimes \cdots \otimes I_2}_{n-k} \in M_2(\mathbb{C})^{\otimes n} = M_{2^n}(\mathbb{C}).$$

Letting $\sigma_I := I_2$, observe that

$$\{\sigma_{W_1}^1 \sigma_{W_2}^2 \cdots \sigma_{W_n}^n \mid W_j \in \{I, X, Y, Z\}\}$$

is a basis of $M_2(\mathbb{C})^{\otimes n}$.

Quantum Max Cut

Pauli matrices

The Pauli matrices are the following three self-adjoint 2×2 matrices

$$\sigma_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{Pauli}$$

For $W \in \{X, Y, Z\}$ and $k, n \in \mathbb{N}$ we shall also use

$$\sigma_W^k = \underbrace{I_2 \otimes \cdots \otimes I_2}_{k-1} \otimes \sigma_W \otimes \underbrace{I_2 \otimes \cdots \otimes I_2}_{n-k} \in M_2(\mathbb{C})^{\otimes n} = M_{2^n}(\mathbb{C}).$$

Letting $\sigma_I := I_2$, observe that

$$\{\sigma_{W_1}^1 \sigma_{W_2}^2 \cdots \sigma_{W_n}^n \mid W_j \in \{I, X, Y, Z\}\}$$

is a basis of $M_2(\mathbb{C})^{\otimes n}$.
Given $i \neq j$, then $\sigma_W^i, \sigma_{W'}^j$ commute:

$$\sigma_W^i \; \sigma_{W'}^j = \sigma_{W'}^j \; \sigma_W^i.$$

Quantum Max Cut (QMC)

QMC Hamiltonian (Pauli Form)

The QMC Hamiltonian of a graph $G = (V, E)$ is given by

$$H_G = \sum_{(i,j)\in E(G)} w_{ij}\left(I - \sigma_X^i\sigma_X^j - \sigma_Y^i\sigma_Y^j - \sigma_Z^i\sigma_Z^j\right) \in M_{2^n}(\mathbb{C})_{\text{sa}}$$

where the σ_W are Pauli matrices and

$$\sigma_W^k = \underbrace{I_2 \otimes \cdots \otimes I_2}_{k-1} \otimes \sigma_W \otimes \underbrace{I_2 \otimes \cdots \otimes I_2}_{n-k}.$$

Quantum Max Cut (QMC)

QMC Hamiltonian (Pauli Form)

The QMC Hamiltonian of a graph $G = (V, E)$ is given by

$$H_G = \sum_{(i,j) \in E(G)} w_{ij} \left(I - \sigma_X^i \sigma_X^j - \sigma_Y^i \sigma_Y^j - \sigma_Z^i \sigma_Z^j \right) \in M_{2^n}(\mathbb{C})_{\mathrm{sa}}$$

where the σ_W are Pauli matrices and

$$\sigma_W^k = \underbrace{I_2 \otimes \cdots \otimes I_2}_{k-1} \otimes \sigma_W \otimes \underbrace{I_2 \otimes \cdots \otimes I_2}_{n-k}.$$

Quantum Max Cut

QMC asks for the biggest eigenvalue of H_G

(and, if possible, the associated eigenvector/state).

Quantum Max Cut (QMC)

Physics motivation

Quantum Max Cut (QMC)

Physics motivation

- exists

Quantum Max Cut (QMC)

Physics motivation

- exists; QMC is a natural maximization variant of the anti-ferromagnetic Heisenberg XYZ model;

- Place a qubit (= vector in $\mathbb{C}^2$) in each vertex;
- Maximize the (2-local) Hamiltonian (= total energy of the system);

- QMC (= a special local Hamiltonian problem) was named by Gharibian & Parekh[2019];

- Max Cut (MC) is NP-hard,
  QMC is a prototype of a QMA-hard problem.

Piddock & Montanaro[2017], Cubitt & Montanaro[2016]

Quantum Max Cut

SWAP operators

$$H_G = \sum_{(i,j)\in E(G)} w_{ij} \left(I - \sigma_X^i \sigma_X^j - \sigma_Y^i \sigma_Y^j - \sigma_Z^i \sigma_Z^j\right)$$

The matrix

$$\mathrm{Swap}_{ij} = \frac{1}{2}(I + \sigma_X^i \sigma_X^j + \sigma_Y^i \sigma_Y^j + \sigma_Z^i \sigma_Z^j)$$

is called a SWAP operator.

Quantum Max Cut

SWAP operators

$$H_G = \sum_{(i,j)\in E(G)} w_{ij}\left(I - \sigma_X^i\sigma_X^j - \sigma_Y^i\sigma_Y^j - \sigma_Z^i\sigma_Z^j\right)$$

The matrix

$$\mathrm{Swap}_{ij} = \frac{1}{2}(I + \sigma_X^i\sigma_X^j + \sigma_Y^i\sigma_Y^j + \sigma_Z^i\sigma_Z^j)$$

is called a SWAP operator.

For instance, if $n = 2$, then

$$\mathrm{Swap}_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Quantum Max Cut

SWAP operators

$$H_G = \sum_{(i,j)\in E(G)} w_{ij} \left(I - \sigma_X^i \sigma_X^j - \sigma_Y^i \sigma_Y^j - \sigma_Z^i \sigma_Z^j\right)$$

The matrix

$$\text{Swap}_{ij} = \frac{1}{2}(I + \sigma_X^i \sigma_X^j + \sigma_Y^i \sigma_Y^j + \sigma_Z^i \sigma_Z^j)$$

is called a SWAP operator.

Thus, we can rewrite the QMC Hamiltonian as

QMC Hamiltonian (SWAP Form)

$$H_G = \sum_{(i,j)\in E(G)} 2 w_{ij} (I - \text{Swap}_{ij})$$

Quantum Max Cut

SWAP operators

The SWAP operator

$$\mathrm{Swap}_{ij} = \frac{1}{2}(I + \sigma_X^i \sigma_X^j + \sigma_Y^i \sigma_Y^j + \sigma_Z^i \sigma_Z^j)$$

sends the rank one tensor

$$v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n \in (\mathbb{C}^2)^{\otimes n}$$

to the rank one tensor

$$v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n \in (\mathbb{C}^2)^{\otimes n},$$

where $v_k \in \mathbb{C}^2$.

Quantum Max Cut

SWAP operators

The SWAP operator

$$\mathrm{Swap}_{ij} = \frac{1}{2}(I + \sigma_X^i \sigma_X^j + \sigma_Y^i \sigma_Y^j + \sigma_Z^i \sigma_Z^j)$$

sends the rank one tensor

$$v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n \in (\mathbb{C}^2)^{\otimes n}$$

to the rank one tensor

$$v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n \in (\mathbb{C}^2)^{\otimes n},$$

where $v_k \in \mathbb{C}^2$.

Let M_n^{Swap} be the SWAP algebra generated by the Swap_{ij} inside $M_{2^n}(\mathbb{C})$.

SWAP operators

$$\mathrm{Swap}_{ij}(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n.$$

Some relations satisfied by the SWAP operators (indices i, j, k, l distinct):

SWAP operators

$$\mathrm{Swap}_{ij}(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n.$$

Some relations satisfied by the SWAP operators (indices i, j, k, l distinct):

$$\left.\begin{aligned} \mathrm{Swap}_{ij}^2 &= I_2, \\ \mathrm{Swap}_{ij}\,\mathrm{Swap}_{jk} &= \mathrm{Swap}_{ik}\,\mathrm{Swap}_{ij}, \\ \mathrm{Swap}_{ij}\,\mathrm{Swap}_{kl} &= \mathrm{Swap}_{kl}\,\mathrm{Swap}_{ij}. \end{aligned}\right\} \text{symmetric group}$$

SWAP operators

$$\mathrm{Swap}_{ij}(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n.$$

Some relations satisfied by the SWAP operators (indices i, j, k, l distinct):

$$\mathrm{Swap}_{ij}^2 = I_2,$$
$$\mathrm{Swap}_{ij}\,\mathrm{Swap}_{jk} = \mathrm{Swap}_{ik}\,\mathrm{Swap}_{ij},$$
$$\mathrm{Swap}_{ij}\,\mathrm{Swap}_{kl} = \mathrm{Swap}_{kl}\,\mathrm{Swap}_{ij}.$$

$$\mathrm{Swap}_{ij}\mathrm{Swap}_{jk} + \mathrm{Swap}_{jk}\mathrm{Swap}_{ij} = \mathrm{Swap}_{ij} + \mathrm{Swap}_{jk} + \mathrm{Swap}_{ik} - I_2$$

SWAP algebra

Symmetric group

$$\mathrm{Swap}_{ij}(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n.$$

Since the transpositions (i, j) generate the symmetric group S_n, the map

$$(i, j) \mapsto \mathrm{Swap}_{ij}$$

gives a representation of the symmetric group S_n on $(\mathbb{C}^2)^{\otimes n}$.

SWAP algebra

Symmetric group

$$\mathrm{Swap}_{ij}(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n.$$

Since the transpositions (i, j) generate the symmetric group S_n, the map

$$(i, j) \mapsto \mathrm{Swap}_{ij}$$

gives a representation of the symmetric group S_n on $(\mathbb{C}^2)^{\otimes n}$.

By Maschke's Theorem (the group algebra $\mathbb{C}S_n$ is semisimple), this representation decomposes into a direct sum of irreps (=irreducible representations).

SWAP algebra

Symmetric group

$$\mathrm{Swap}_{ij}(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n.$$

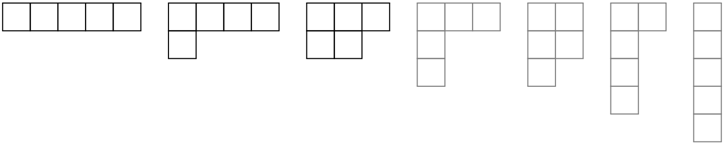
Since the transpositions (i, j) generate the symmetric group S_n, the map

$$(i, j) \mapsto \mathrm{Swap}_{ij}$$

gives a representation of the symmetric group S_n on $(\mathbb{C}^2)^{\otimes n}$.

By Maschke's Theorem (the group algebra $\mathbb{C}S_n$ is semisimple), this representation decomposes into a direct sum of irreps (=irreducible representations).

It is well known that the irreps of the symmetric group S_n are indexed by partitions λ of n, or equivalently, Young diagrams: $\ldots \mathscr{S}_\lambda$



SWAP algebra

Schur-Weyl duality

$$\mathrm{Swap}_{ij}(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n.$$

$\mathrm{GL}_2(\mathbb{C})$ also acts on $(\mathbb{C}^2)^{\otimes n}$:

$$g \cdot (v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n.$$

SWAP algebra

Schur-Weyl duality

$$\mathrm{Swap}_{ij}(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n.$$

$\mathrm{GL}_2(\mathbb{C})$ also acts on $(\mathbb{C}^2)^{\otimes n}$:

$$g \cdot (v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n.$$

- This action commutes with the action of the SWAP operators:

$$\mathrm{Swap}_{ij} \circ g = g \circ \mathrm{Swap}_{ij}.$$

SWAP algebra

Schur-Weyl duality

$$\mathrm{Swap}_{ij}(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n.$$

$\mathrm{GL}_2(\mathbb{C})$ also acts on $(\mathbb{C}^2)^{\otimes n}$:

$$g \cdot (v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n.$$

- This action commutes with the action of the SWAP operators:

$$\mathrm{Swap}_{ij} \circ g = g \circ \mathrm{Swap}_{ij}.$$

- Irreps of $\mathrm{GL}_2(\mathbb{C})$ are indexed by two row Young diagrams with an arbitrary number of boxes. $\ldots \mathscr{L}_{[n-k,k]}$

SWAP algebra

Schur-Weyl duality

$$\mathrm{Swap}_{ij}(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n.$$

$\mathrm{GL}_2(\mathbb{C})$ also acts on $(\mathbb{C}^2)^{\otimes n}$:

$$g \cdot (v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n.$$

Theorem (Schur-Weyl duality)

The space $(\mathbb{C}^2)^{\otimes n}$ decomposes under the action of $\mathrm{GL}_2(\mathbb{C}) \times S_n$ as

$$(\mathbb{C}^2)^{\otimes n} \cong \bigoplus_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathscr{L}_{[n-k,k]} \otimes \mathscr{S}_{[n-k,k]}.$$

In particular, as S_n-module (or SWAP algebra-module),

$$(\mathbb{C}^2)^{\otimes n} \cong \bigoplus_{k=0}^{\lfloor \frac{n}{2} \rfloor} (\mathscr{S}_{[n-k,k]})^{\dim \mathscr{L}_{[n-k,k]}}.$$

SWAP algebra

Schur-Weyl duality (cont'd)

$$\mathrm{Swap}_{ij}(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n.$$

Corollary

The Swap Matrix Algebra M_n^{Swap} is the direct sum of simple algebras generated by the two row irreps of the symmetric group S_n:

$$M_n^{\mathrm{Swap}} \cong \bigoplus_{k=0}^{\lfloor \frac{n}{2} \rfloor} M_{\dim \mathscr{S}_{[n-k,k]}}(\mathbb{C})$$

$\dim M_n^{\mathrm{Swap}} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{n-2k+1}{n-k+1} \binom{n}{k} \right)^2 = \frac{1}{n+1} \binom{2n}{n}$ is the n-th Catalan number C_n.

SWAP algebra

Schur-Weyl duality (cont'd)

$$\mathrm{Swap}_{ij}(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n.$$

Corollary

The Swap Matrix Algebra M_n^{Swap} is the direct sum of simple algebras generated by the two row irreps of the symmetric group S_n:

$$M_n^{\mathrm{Swap}} \cong \bigoplus_{k=0}^{\lfloor \frac{n}{2} \rfloor} M_{\dim \mathscr{S}_{[n-k,k]}}(\mathbb{C})$$

Theorem

The Swap Matrix Algebra M_n^{Swap} is given by the following presentation: $M_n^{\mathrm{Swap}} \cong \mathbb{C}\langle \mathrm{swap}_{ij} \rangle / \mathcal{J}_{\mathrm{Swap}}$*, where $\mathcal{J}_{\mathrm{Swap}}$ is the ideal generated by*

$$\begin{aligned}
\mathrm{swap}_{ij}^2 &= I, \\
\mathrm{swap}_{ij}\,\mathrm{swap}_{jk} &= \mathrm{swap}_{ik}\,\mathrm{swap}_{ij}, \\
\mathrm{swap}_{ij}\,\mathrm{swap}_{kl} &= \mathrm{swap}_{kl}\,\mathrm{swap}_{ij}, \\
\mathrm{swap}_{ij}\mathrm{swap}_{jk} + \mathrm{swap}_{jk}\mathrm{swap}_{ij} &= \mathrm{swap}_{ij} + \mathrm{swap}_{jk} + \mathrm{swap}_{ik} - I.
\end{aligned}$$

$\dim M_n^{\mathrm{Swap}} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{n-2k+1}{n-k+1} \binom{n}{k} \right)^2 = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number C_n.

Efficient Approximations to QMC

Helton-McCullough Positivstellensatz made effective aka nc Lasserre hierarchy

To $h \in \mathbb{C}\langle \text{swap} \rangle$ let

$$\nu_d(h) := \min \{\nu \mid \nu - h \in \text{SOS}_{2d} + \mathcal{J}_{\text{Swap}}\},$$

where SOS_{2d} denotes the set of all sums of squares of polynomials in the free nc variables swap_{ij}, each having degree $\leq d$.

Efficient Approximations to QMC

Helton-McCullough Positivstellensatz made effective aka nc Lasserre hierarchy

To $h \in \mathbb{C}\langle \text{swap} \rangle$ let

$$\nu_d(h) := \min \{ \nu \mid \nu - h \in \text{SOS}_{2d} + \mathcal{J}_{\text{Swap}} \},$$

where SOS_{2d} denotes the set of all sums of squares of polynomials in the free nc variables swap_{ij}, each having degree $\leq d$.

- $\nu_d(h) \geq \text{eig}_{\max} h(\text{Swap})$

Efficient Approximations to QMC

Helton-McCullough Positivstellensatz made effective aka nc Lasserre hierarchy

To $h \in \mathbb{C}\langle \text{swap} \rangle$ let

$$\nu_d(h) := \min \{ \nu \mid \nu - h \in \text{SOS}_{2d} + \mathcal{J}_{\text{Swap}} \},$$

where SOS_{2d} denotes the set of all sums of squares of polynomials in the free nc variables swap_{ij}, each having degree $\leq d$.

- $\nu_d(h) \geq \text{eig}_{\max} h(\text{Swap})$

- $\nu_{\lceil n/2 \rceil}(h) = \text{eig}_{\max} h(\text{Swap})$

Efficient Approximations to QMC

Helton-McCullough Positivstellensatz made effective aka nc Lasserre hierarchy

To $h \in \mathbb{C}\langle \text{swap} \rangle$ let

$$\nu_d(h) := \min \{ \nu \mid \nu - h \in \text{SOS}_{2d} + \mathcal{J}_{\text{Swap}} \},$$

where SOS_{2d} denotes the set of all sums of squares of polynomials in the free nc variables swap_{ij}, each having degree $\leq d$.

Veroneses are column vectors $V_d(n)$, which consist of degree d monomials in the $n(n-1)/2$ variables $\text{swap}_{ij},\ i < j$, ordered w.r.t. grlex.

Efficient Approximations to QMC

Helton-McCullough Positivstellensatz made effective aka nc Lasserre hierarchy

To $h \in \mathbb{C}\langle \text{swap} \rangle$ let

$$\nu_d(h) := \min \{ \nu \mid \nu - h \in \text{SOS}_{2d} + \mathcal{J}_{\text{Swap}} \},$$

where SOS_{2d} denotes the set of all sums of squares of polynomials in the free nc variables swap_{ij}, each having degree $\leq d$.

Veroneses are column vectors $V_d(n)$, which consist of degree d monomials in the $n(n-1)/2$ variables $\text{swap}_{ij},\ i < j$, ordered w.r.t. grlex.

Lemma

Let $h \in \mathbb{C}\langle \text{swap} \rangle$. Then $h \in \text{SOS}_{2d} + \mathcal{J}_{\text{Swap}}$ iff there is a PsD matrix Γ such that

$$h - V_d(n)^* \Gamma V_d(n) \in \mathcal{J}_{\text{Swap}}.$$

Finding such a Γ can be done with a semidefinite program (SDP).

Given a "good" generating set (e.g., a Gröbner basis) for $\mathcal{J}_{\text{Swap}}$.

QMC

nc Lasserre relaxations (cont'd)

$$\nu_d(h) = \min\left\{\nu \mid \nu - h \in \mathrm{SOS}_{2d} + \mathcal{J}_{\mathrm{Swap}}\right\},$$

$$\begin{aligned} \reflectbox{ν}_d(h) = \max\ & L(h) \\ \text{s.t. } & L \in (\mathrm{SOS}_{2d} + \mathcal{J}_{\mathrm{Swap}})^{\vee} \\ & L(1) = 1. \end{aligned}$$

Here $(\mathrm{SOS}_{2d} + \mathcal{J}_{\mathrm{Swap}})^{\vee}$ denotes the dual cone to the cone $\mathrm{SOS}_{2d} + \mathcal{J}_{\mathrm{Swap}}$,

$$\begin{aligned} (\mathrm{SOS}_{2d} + \mathcal{J}_{\mathrm{Swap}})^{\vee} = \Big\{ & L : \mathbb{C}\langle \mathrm{swap}\rangle_{2d} \to \mathbb{C} \mid L \text{ linear with } L(\mathrm{SOS}_{2d}) \subseteq \mathbb{R}_{\geq 0}, \\ & L(\mathcal{J}_{\mathrm{Swap}} \cap \mathbb{C}\langle \mathrm{swap}\rangle_{2d}) = \{0\} \Big\}. \end{aligned}$$

QMC

nc Lasserre relaxations (cont'd)

$$\nu_d(h) = \min \left\{ \nu \mid \nu - h \in \mathrm{SOS}_{2d} + \mathcal{J}_{\mathrm{Swap}} \right\},$$

$$\begin{aligned} \rotatebox[origin=c]{180}{ν}_d(h) = \max\ & L(h) \\ \text{s.t. } & L \in (\mathrm{SOS}_{2d} + \mathcal{J}_{\mathrm{Swap}})^{\vee} \\ & L(1) = 1. \end{aligned}$$

Here $(\mathrm{SOS}_{2d} + \mathcal{J}_{\mathrm{Swap}})^{\vee}$ denotes the dual cone to the cone $\mathrm{SOS}_{2d} + \mathcal{J}_{\mathrm{Swap}}$.

This is another SDP.

- (strong duality) $\rotatebox[origin=c]{180}{ν}_d(h) = \nu_d(h)$.
- (pseudomoments) Implement $\rotatebox[origin=c]{180}{ν}_d(h)$ with the help of moment matrices.

QMC

nc Lasserre relaxations – example

$$\nu_d(h) = \min\left\{\nu \mid \nu - h \in \mathrm{SOS}_{2d} + \mathscr{J}_{\mathrm{Swap}}\right\},$$
$$\lambda_d(h) = \max\left\{L(h) \mid L \in (\mathrm{SOS}_{2d} + \mathscr{J}_{\mathrm{Swap}})^{\vee},\, L(1) = 1\right\}.$$

Take $n = 3$, $d = 1$. Then $V_1(3) = (1, s_{12}, s_{13}, s_{23})^*$.

For brevity we use s_{ij} for swap_{ij} here.

The symbolic Hankel matrix is

$$\mathcal{M}_1(3) = V_1(3)V_1(3)^* = \begin{bmatrix} 1 & s_{12} & s_{13} & s_{23} \\ s_{12} & s_{12}^2 & s_{12}s_{13} & s_{12}s_{23} \\ s_{13} & s_{13}s_{12} & s_{13}^2 & s_{13}s_{23} \\ s_{23} & s_{23}s_{12} & s_{23}s_{13} & s_{23}^2 \end{bmatrix}$$

and the pseudomoments of $L \in (\mathrm{SOS}_{2d} + \mathscr{J}_{\mathrm{Swap}})^{\vee}$ are

$$\mathcal{M}_1(L) = \begin{bmatrix} L(1) & L(s_{12}) & L(s_{13}) & L(s_{23}) \\ L(s_{12}) & L(s_{12}^2) & L(s_{12}s_{13}) & L(s_{12}s_{23}) \\ L(s_{13}) & L(s_{13}s_{12}) & L(s_{13}^2) & L(s_{13}s_{23}) \\ L(s_{23}) & L(s_{23}s_{12}) & L(s_{23}s_{13}) & L(s_{23}^2) \end{bmatrix}$$

QMC

nc Lasserre relaxations – example (cont'd)

$$n = 3,\ d = 1,\ V_1(3) = (1, s_{12}, s_{13}, s_{23})^*$$

The space of quadratics in the SWAPs is spanned by the entries of $V_1(3)$ together with one element, e.g., $s_{12}s_{13}$.

QMC

nc Lasserre relaxations – example (cont'd)

$$n = 3,\ d = 1,\ V_1(3) = (1, s_{12}, s_{13}, s_{23})^*$$

The space of quadratics in the SWAPs is spanned by the entries of $V_1(3)$ together with one element, e.g., $s_{12}s_{13}$. Namely,

$$\begin{aligned} s_{ij}^2 &= 1 & s_{12}s_{23} &= -1 + s_{12} + s_{13} + s_{23} - s_{12}s_{13} \\ s_{13}s_{23} &= s_{12}s_{13} & s_{13}s_{12} &= -1 + s_{12} + s_{13} + s_{23} - s_{12}s_{13} \\ s_{23}s_{12} &= s_{12}s_{13} & s_{23}s_{13} &= -1 + s_{12} + s_{13} + s_{23} - s_{12}s_{13} \end{aligned}$$

QMC

nc Lasserre relaxations – example (cont'd)

$$n = 3,\ d = 1,\ V_1(3) = (1, s_{12}, s_{13}, s_{23})^*$$

The space of quadratics in the SWAPs is spanned by the entries of $V_1(3)$ together with one element, e.g., $s_{12}s_{13}$. Namely,

$$\begin{aligned} s_{ij}^2 &= 1 & s_{12}s_{23} &= -1 + s_{12} + s_{13} + s_{23} - s_{12}s_{13} \\ s_{13}s_{23} &= s_{12}s_{13} & s_{13}s_{12} &= -1 + s_{12} + s_{13} + s_{23} - s_{12}s_{13} \\ s_{23}s_{12} &= s_{12}s_{13} & s_{23}s_{13} &= -1 + s_{12} + s_{13} + s_{23} - s_{12}s_{13} \end{aligned}$$

With this the pseudomoments of $L \in (\mathrm{SOS}_{2d} + \mathcal{J}_{\mathrm{Swap}})^\vee$ simplify

$$\begin{aligned} \mathcal{M}_1(L) &= \begin{bmatrix} L(1) & L(s_{12}) & L(s_{13}) & L(s_{23}) \\ L(s_{12}) & L(s_{12}^2) & L(s_{12}s_{13}) & L(s_{12}s_{23}) \\ L(s_{13}) & L(s_{13}s_{12}) & L(s_{13}^2) & L(s_{13}s_{23}) \\ L(s_{23}) & L(s_{23}s_{12}) & L(s_{23}s_{13}) & L(s_{23}^2) \end{bmatrix} \\ &= \begin{bmatrix} 1 & \ell_{12} & \ell_{13} & \ell_{23} \\ \ell_{12} & 1 & q & -1 + \ell_{12} + \ell_{13} + \ell_{23} - q \\ \ell_{13} & q^* & 1 & q \\ \ell_{23} & -1 + \ell_{12} + \ell_{13} + \ell_{23} - q^* & q^* & 1 \end{bmatrix}, \end{aligned}$$

where $\ell_{ij} = L(s_{ij})$ and $q = L(s_{12}s_{13})$.

QMC

nc Lasserre relaxations (cont'd)

$$\nu_d(h) = \min\left\{\nu \mid \nu - h \in \mathrm{SOS}_{2d} + \mathscr{J}_{\mathrm{Swap}}\right\},$$
$$\reflectbox{ν}_d(h) = \max\left\{L(h) \mid L \in (\mathrm{SOS}_{2d} + \mathscr{J}_{\mathrm{Swap}})^{\vee},\, L(1) = 1\right\}.$$

We can now rewrite $\reflectbox{$\nu$}_d(h)$ as an SDP as follows:

$$\begin{aligned} \reflectbox{ν}_d(h) = \max\ & \langle \mathcal{M}_d(L), \Gamma_h \rangle \\ \text{s.t. } & \mathcal{M}_d(L) \succeq 0 \\ & \mathcal{M}_d(L)_{1,1} = 1 \\ & L(\mathscr{J}_{\mathrm{Swap}} \cap \mathbb{C}\langle \mathrm{swap} \rangle_{2d}) = \{0\}, \end{aligned}$$

where Γ_h is a (not necessarily positive semidefinite) Gram matrix for h,

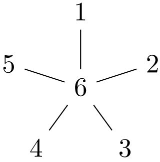
$$h = V_d(n)^* \Gamma_h V_d(n).$$

QMC

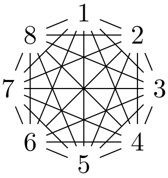
Numerical results

Takahashi, Rayudu, Zhou, King, Thompson, Parekh[2023] give many examples of the 1st nc Lasserre hierarchy.

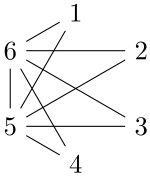
- It is exact for



star graphs



even cliques



certain crown graphs

- It is non-exact for odd cliques, and many small ($n \leq 6$) graphs.

QMC

Numerical examples

The second nc Moment-SOS SDP relaxation for QMC has size

$$1 + \binom{n}{2} + \binom{n}{3} + 3\binom{n}{4} = \frac{1}{24}\left(3n^4 - 14n^3 + 33n^2 - 22n + 24\right)$$

n	1	2	3	4	5	6	7	8	9	10	12	15	20
size	1	2	5	14	36	81	162	295	499	796	1772	4656	15866

QMC

Numerical examples

The second nc Moment-SOS SDP relaxation for QMC has size

$$1 + \binom{n}{2} + \binom{n}{3} + 3\binom{n}{4} = \frac{1}{24}\left(3n^4 - 14n^3 + 33n^2 - 22n + 24\right)$$

Proposition

For $n \leq 8$ the second nc Moment-SOS SDP relaxation for QMC of an n vertex QMC with uniform edge weights is up to the tolerance of 10^{-7} exact, i.e., equal to the true max.

Uses nc Gröbner bases.

QMC

Numerical examples

The second nc Moment-SOS SDP relaxation for QMC has size

$$1 + \binom{n}{2} + \binom{n}{3} + 3\binom{n}{4} = \frac{1}{24}\left(3n^4 - 14n^3 + 33n^2 - 22n + 24\right)$$

Proposition

For $n \leq 8$ the second nc Moment-SOS SDP relaxation for QMC of an n vertex QMC with uniform edge weights is up to the tolerance of 10^{-7} exact, i.e., equal to the true max. Uses nc Gröbner bases.

- ? It would be interesting to find the smallest graph on which the second relaxation is not exact.
- i It *appears* that the first classical relaxation is worse than the quantum one for swaps.

QMC

Rounding aka solving a moment problem approximately

$$\mathrm{eig}_{\max}(H) = \langle Hv, v\rangle = \mathrm{tr}(H \underbrace{vv^T}_{\rho}), \quad \rho \text{ is a state}$$

- Round SDP solutions to product states $\rho = \rho_1 \otimes \cdots \otimes \rho_n$
  Brandao & Harrow[2016], Bravyi & Gosset & König & Temme[2019], Gharibian & Parekh[2019], Parekh & Thompson[2021];
- Parekh & Thompson[2022]: "optimal" rounding to product state = $1/2-$approximation;
- Anshu & Gosset & Morenz[2020]: $0.531-$approximation;
- Parekh & Thompson[2021]: $0.533-$approximation;
- King[2023]: $0.582-$approximation;
- Hwang & Neeman & Parekh & Thompson & Wright[2023]: Unique Games hardness of $(0.956+\varepsilon)-$approximation for QMC, assuming a plausible conjecture in Gaussian geometry;

QMC

Rounding ala Parekh & Thompson^{2021}: 0.533−approximation

Algorithm 1 PT2021 Approximation Algorithm for QMC

1. Input graph $G = (V, E)$ with weights $w = \{w_e \geq 0\}_{e \in E}$, solve 1st nc Lasserre. Let the matrix $\mathcal{M}$ be an optimal solution.
2. For each $(i, j) \in E$ calculate $x_{ij} := [1 - 2\mathcal{M}(\mathrm{Swap}_{ij}, 1)]/3$.
3. Pick $d \in \mathbb{N}$, and define $L := \{e \in E \mid x_e > \alpha(d) := \frac{d+3}{3(d+1)}\}$. Find a maximum-weight matching F in the graph $G_L := (V, L)$ w.r.t weights $\{w_e\}_{e \in L}$. Let U be the vertices unmatched by F.
4. Define a quantum state:
$$\rho_F := \prod_{ij \in F} \left(\frac{\mathbb{I} - \mathrm{Swap}_{ij}}{2} \right) \prod_{v \in U} \frac{I_2}{2}. \quad (1)$$
5. Find the optimal product state ρ_{PS}.
6. Output the better of ρ_F and ρ_{PS}.

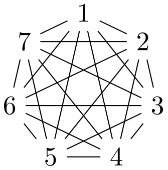
QMC

Exact solutions – clique

$$H_G = \sum_{(i,j)\in E(G)} 2(I - \mathrm{Swap}_{ij}), \qquad M_n^{\mathrm{Swap}} \cong \bigoplus_{k=0}^{\lfloor \frac{n}{2} \rfloor} M_{\dim \mathscr{S}_{[n-k,k]}}(\mathbb{C})$$

Example

Let $G = K_n$ be the clique on n vertices. Then



$$H_{K_n} = 2\sum_{i<j}(I - \mathrm{Swap}_{ij}).$$

- Under each irrep λ, $H_{K_n}^{\lambda}$ is a scalar matrix.

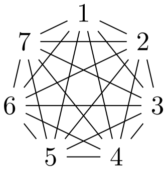
QMC

Exact solutions – clique

$$H_G = \sum_{(i,j)\in E(G)} 2(I - \mathrm{Swap}_{ij}), \qquad M_n^{\mathrm{Swap}} \cong \bigoplus_{k=0}^{\lfloor \frac{n}{2} \rfloor} M_{\dim \mathscr{S}_{[n-k,k]}}(\mathbb{C})$$

Example

Let $G = K_n$ be the clique on n vertices. Then



$$H_{K_n} = 2\sum_{i<j}(I - \mathrm{Swap}_{ij}).$$

- Under each irrep λ, $H_{K_n}^{\lambda}$ is a scalar matrix.
- $H_{K_n}^{[n-k,k]} = \binom{n}{2} + k^2 - k(n+1)$ (hook length & Murnaghan-Nakayama rule).

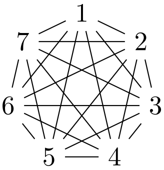
QMC

Exact solutions – clique

$$H_G = \sum_{(i,j)\in E(G)} 2(I - \mathrm{Swap}_{ij}), \qquad M_n^{\mathrm{Swap}} \cong \bigoplus_{k=0}^{\lfloor \frac{n}{2} \rfloor} M_{\dim \mathscr{S}_{[n-k,k]}}(\mathbb{C})$$

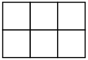
Example

Let $G = K_n$ be the clique on n vertices. Then



$$H_{K_n} = 2\sum_{i<j}(I - \mathrm{Swap}_{ij}).$$

- Under each irrep λ, $H_{K_n}^{\lambda}$ is a scalar matrix.
- $H_{K_n}^{[n-k,k]} = \binom{n}{2} + k^2 - k(n+1)$ (hook length & Murnaghan-Nakayama rule).
- QMC value of K_n is the max of $H_{K_n}^{[n-k,k]}$ for $k = 0, \ldots, \lfloor \frac{n}{2} \rfloor$, and is attained at $k = \lfloor \frac{n}{2} \rfloor$.

 or 

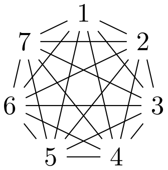
QMC

Exact solutions – clique

$$H_G = \sum_{(i,j)\in E(G)} 2(I - \mathrm{Swap}_{ij}), \qquad M_n^{\mathrm{Swap}} \cong \bigoplus_{k=0}^{\lfloor \frac{n}{2} \rfloor} M_{\dim \mathscr{S}_{[n-k,k]}}(\mathbb{C})$$

Example

Let $G = K_n$ be the clique on n vertices. Then



$$H_{K_n} = 2\sum_{i<j}(I - \mathrm{Swap}_{ij}).$$

- Under each irrep λ, $H_{K_n}^{\lambda}$ is a scalar matrix.
- $H_{K_n}^{[n-k,k]} = \binom{n}{2} + k^2 - k(n+1)$ (hook length & Murnaghan-Nakayama rule).

ℹ This allows us to write an nc Moment-SOS SDP relaxation scheme for optimizing H_G^{λ} inside a two row irrep λ.

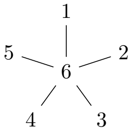
QMC

Exact solutions – star graph

$$H_G = \sum_{(i,j)\in E(G)} 2(I - \mathrm{Swap}_{ij}), \qquad M_n^{\mathrm{Swap}} \cong \bigoplus_{k=0}^{\lfloor \frac{n}{2} \rfloor} M_{\dim \mathscr{S}_{[n-k,k]}}(\mathbb{C})$$

Example

Let $G = \bigstar_n$ be the star graph on n vertices. Then



$$H_{\bigstar_n} = 2 \sum_{j<n} (I - \mathrm{Swap}_{jn}).$$

- $H_{\bigstar_n}^{[n-k,k]}$ has two eigenvalues, namely $2(n-k+1) > 2k$.

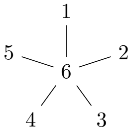
QMC

Exact solutions – star graph

$$H_G = \sum_{(i,j)\in E(G)} 2(I - \mathrm{Swap}_{ij}), \qquad M_n^{\mathrm{Swap}} \cong \bigoplus_{k=0}^{\lfloor \frac{n}{2} \rfloor} M_{\dim \mathscr{S}_{[n-k,k]}}(\mathbb{C})$$

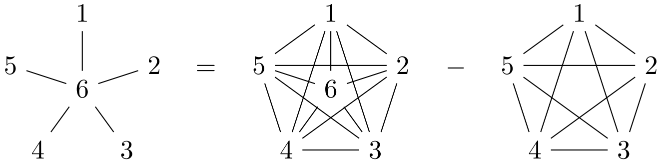
Example

Let $G = \bigstar_n$ be the star graph on n vertices. Then



$$H_{\bigstar_n} = 2\sum_{j<n}(I - \mathrm{Swap}_{jn}).$$

- $H_{\bigstar_n}^{[n-k,k]}$ has two eigenvalues, namely $2(n-k+1) > 2k$.



branching rule

QMC

Exact solutions – star graph (cont'd)

$\bigstar_n = K_n - K_{n-1}$

$$H_{\bigstar_n}^{[n-k,k]} = H_{K_n}^{[n-k,k]} - H_{K_{n-1}}^{[n-k,k]}.$$

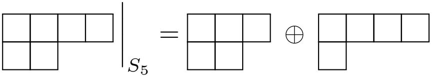
QMC

Exact solutions – star graph (cont'd)

$\bigstar_n = K_n - K_{n-1}$

$$H^{[n-k,k]}_{\bigstar_n} = H^{[n-k,k]}_{K_n} - H^{[n-k,k]}_{K_{n-1}}.$$

Branching rule:



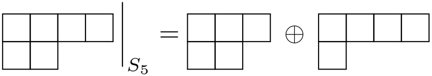
QMC

Exact solutions – star graph (cont'd)

$\bigstar_n = K_n - K_{n-1}$

$$H_{\bigstar_n}^{[n-k,k]} = H_{K_n}^{[n-k,k]} - H_{K_{n-1}}^{[n-k,k]}.$$

Branching rule:



$$\begin{aligned} H_{\bigstar_n}^{[n-k,k]} &= H_{K_n}^{[n-k,k]} - H_{K_{n-1}}^{[n-k,k]} \\ &= H_{K_n}^{[n-k,k]} - \left(H_{K_{n-1}}^{[n-k-1,k]} \oplus H_{K_{n-1}}^{[n-k,k-1]} \right). \end{aligned}$$

QMC

Tree-clique decomposition

For any connected graph G, the tree clique decomposition of G, denoted $\mathcal{T}(G)$, consists of a rooted tree $T = \{v_1, \ldots, v_m\}$, and connected graphs $\{G(v_1) = G, \ldots, G(v_m)\}$ such that:

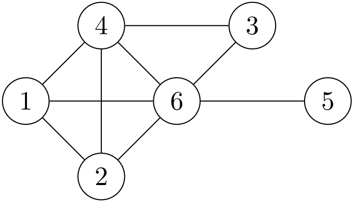
- For any vertex v_i of T which is not a leaf vertex, let $c_1, \ldots, c_k$ be its children. Then

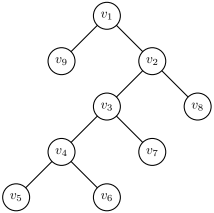
$$G(v_i)^c = \dot{\bigcup_{j \in \{1,2,\ldots,k\}}} G(c_j),$$

- For any leaf vertex v_j of T we have that $G(v_j)^c$ is connected or $G(v_j)^c$ is totally disconnected.

QMC

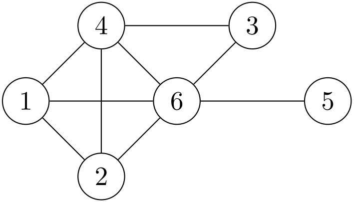
Tree-clique decomposition – example

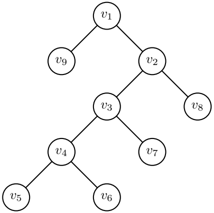


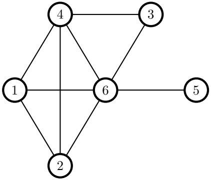


QMC

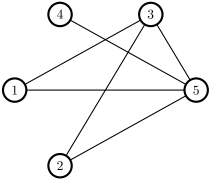
Tree-clique decomposition – example



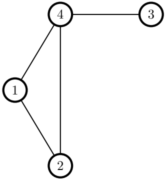




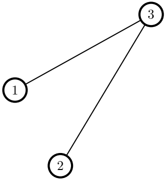
$G(v_1)$



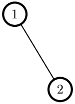
$G(v_2)$



$G(v_3)$



$G(v_4)$



$G(v_5)$

$G(v_6)$

$G(v_7)$

$G(v_8)$

$G(v_9)$

Theorem

Let G be a graph and $\mathcal{T}(G) = \{T, \{G(v_1), \ldots, G(v_m)\}\}$ be its tree-clique decomposition. Then

- *For any vertex $v \in T$ with children $c_1, \ldots, c_k$,*

$$H_{G(v)} = H_{K(G(v))} - \sum_{j \in \{1,\ldots,k\}} H_{G(c_j)}$$

- *Let L denote the set of leaf vertices in T, and R be all non-leaf vertices. Let $d(v)$ denote the depth of vertex v in the the tree, with root $d(v_1) = 0$. Then*

$$H_G = \sum_{r \in R} (-1)^{d(r)} H_{K(G(r))} + \sum_{l \in L} (-1)^{d(l)} H_{G(l)}$$

Theorem

Let G be a graph and $\mathcal{T}(G) = \{T, \{G(v_1), \ldots, G(v_m)\}\}$ be its tree-clique decomposition. Then

- *For any vertex $v \in T$ with children $c_1, \ldots, c_k$,*

$$H_{G(v)} = H_{K(G(v))} - \sum_{j \in \{1,\ldots,k\}} H_{G(c_j)}$$

- *Let L denote the set of leaf vertices in T, and R be all non-leaf vertices. Let $d(v)$ denote the depth of vertex v in the the tree, with root $d(v_1) = 0$. Then*

$$H_G = \sum_{r \in R} (-1)^{d(r)} H_{K(G(r))} + \sum_{l \in L} (-1)^{d(l)} H_{G(l)}$$

☞ *Given* min and max eigenvalues under all two row irreps of $G(l)$ for every leaf vertex l of T, one can *inductively compute* min and max eigenvalues under all two row irreps of G.

Outro

Quantum Max Cut (QMC) is fun 😎

- ✓ QMC Hamiltonian expressed in terms of SWAP operators
  Identify the SWAP algebra via Schur-Weyl duality

- ✓ Nc Lasserre's relaxation (= Helton-McCullough Positivstellensatz) produces a Moment-SOS SDP hierarchy for QMC

- ✓ Exact solutions for various simple graphs

- ✓ Tree-clique decomposition algorithm for writing a graph as a sum of ±cliques
  Yields a recursive algorithm for solving QMC exactly

- ✓ $\mathbb{C}^2 \rightsquigarrow \mathbb{C}^d$: qudits instead of qubits

¿ Any ideas/thoughts for a better rounding algorithm ?

THE END