Polynomial Lemniscates and Torsional Rigidity (Bergman Space Approximations)

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Joint work with Brian Simanek

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Torsional Rigidity

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- There are varying types of mechanical stress in *elasticity theory*, from compressibility, to tensile strength, shear strain, and torsional rigidity.
- The *torsional rigidity* of an object is its resistance to the twisting force known as torque.

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- We will be interested in how geometry influences torsional rigidity.

Torsional Rigidity

- In 1948 George Pólya conjectured that for any $n \in \mathbb{N}$, among all n-sided polygonal cross-sections with fixed area, the n-gon with maximal torsional rigidity is the regular n-gon.

Calculating Torsional Rigidity

- For a simply connected Jordan domain $\Omega \subseteq \mathbb{C}$, the torsional rigidity $\rho(\Omega)$ of an infinite beam with cross-section Ω is given by

$$\rho(\Omega) := \sup_{u \in C_0^1(\overline{\Omega})} \frac{4\left(\int_\Omega u(z)dA(z)\right)^2}{\int_\Omega |\nabla u(z)|^2 dA(z)}$$

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• The function $\nu(z)$ which attains such a maximum is known as the *stress function* of the region Ω.

Calculating Torsional Rigidity

The stress function for Ω, $\nu(z)$, is a solution to the boundary value problem:

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$$\begin{cases} \Delta\nu & = & -2 \\ \nu\big|_{\delta\Omega} & = & 0 \end{cases}$$

and

$$\rho(\Omega) = 2\int_{\Omega} \nu(z)\,dA(z)$$

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$$\rho(\Omega) = \sigma^2(\Omega).$$

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Region Ω	$\rho(\Omega)$	variables
Disk	$\frac{1}{2}\pi r^4$	r radius
Ellipse	$\frac{\pi a^3 b^3}{a^2 + b^2}$	a, b radii, $a > b$
Square	$\frac{9}{4}\left(\frac{a}{2}\right)^4$	a side length
Equilateral Triangle	$\frac{a^4\sqrt{3}}{80}$	a side length
Rectangle	$\frac{ab^3}{8}\left[\frac{16}{3} - 3.36\frac{b}{a}\left(1 - \frac{b^4}{12a^4}\right)\right]$	a, b side lengths, $a \geq b$

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Let Ω be a bounded finitely connected domain. Then $f(z)$ is the projection of $\bar{z}$ onto $A^2(\Omega)$ if and only if $|z|^2 = F(z) + \overline{F(z)}$ on $\Gamma = \delta\Omega$, where $F'(z) = f(z)$.

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In other words, the Bergman projection of $\bar{z}$ on Ω is the derivative of a function whose real part is $|z|^2/2$ on the boundary of that domain.

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- The Bergman analytic content method implies several continuity properties.
- Regions which are sufficiently ‘similar’ to one another must have nearly equal torsional rigidities.

The Road Map

- To find a region Ω on which you can calculate $\rho(\Omega)$ exactly, choose a function F and examine the lemniscate where

$$F(z) + \overline{F(z)} = |z|^2.$$

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- The difficulty in applying this method lies in verifying that the lemniscate is indeed a simply connected Jordan region.

The Road Map

• One can find examples of regions where one can calculate $\sigma(\Omega)$ exactly by considering regions whose boundary is a bounded connected component of the set

$$\tilde{\Gamma}_F := \{z : \mathrm{Re}[F(z)] = |z|^2/2\},$$

as long as F is holomorphic on the region bounded by $\tilde{\Gamma}_F$.

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as long as F is holomorphic on the region bounded by $\tilde{\Gamma}_F$.

- Since $\sigma(\Omega)$ is calculated using $F'(z)$, not $F(z)$, without loss of generality we may consider

$$\Gamma_F := \{z : \mathrm{Re}[F(z) + k] = |z|^2\}$$

First Results (details)

- The single monomial case with $k = 1$, that is, on domains defined by

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• The best approximation to $\bar{z}$ is the function

$$f(z) = \frac{1}{2}Cnz^{n-1}.$$

• There are values of C such that the set includes no bounded components. Fleeman and Lundberg showed that a bounded connected component exists whenever

$$C \leq \frac{2(n-2)^{\frac{n-2}{2}}}{n^{\frac{n}{2}}}$$

First Results

- We improve this result as follows:

Theorem (K. & Simanek, arXiv preprint)

For $n \geq 3$, $k > 0$ *the set* $\{z : C\,Re[z^n] - |z|^2 + k = 0\}$ *has exactly one bounded component whenever*

$$|C| \leq C^*$$

where

$$C^* := \frac{2k}{n-2}\left(\frac{n-2}{nk}\right)^{n/2}.$$

Further, if $|C| > C^*$, *then the set does not include a bounded component.*

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Theorem (K. & Simanek, arXiv preprint)

If $C, k > 0$ and $n \geq 3$, the set

$$\{z : Re\left[Cz^n + z\right] - |z|^2 + k = 0\}$$

has exactly one bounded connected component if and only if $C \leq C^*$ where

$$C^* := \frac{(2n-4)^n\left(4k(n-2)+\left((n-1)+\sqrt{(n-1)^2+4nk(n-2)}\right)\right)}{2(n-2)^2\left((n-1)+\sqrt{(n-1)^2+4nk(n-2)}\right)^n}$$

First Results

- We may define more general binomial functions,

$$f_{n,j,C,k}(r,\theta) := Cr^n cos(n\theta) + r^j cos(j\theta) - r^2 + k.$$

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- We would like to determine what conditions on n, j, C, k ensure the existence of a bounded connected component for the set,

$$\Gamma_{n,j}(C,k) := \{re^{i\theta} : f_{n,j,C,k}(r,\theta) = 0\}$$

First Results

Theorem (K. & Simanek, arXiv preprint)

If $C, k > 0$ and $n > j > 2$ are natural numbers, then the set $\Gamma_{n,j}(C, k)$ has at least one bounded connected component if and only if $C < C^*$ and $k < k^*$ where

$$C^* := \max_{r \in (0,\infty)} \frac{r^2 - r^j - k}{r^n}$$

and

$$k^* := \left(1 - \frac{2}{j}\right)\left(\frac{2}{j}\right)^{2/(j-2)}.$$

Key Lemma

Lemma (K. & Simanek, arXiv preprint)

Consider $C, k > 0$ *and* $n, j \in \mathbb{N}$ *with* $j < n$. *If the set*

$$\Gamma_{n,j}(C,k) = \{z : Re\left[Cz^n + z^j\right] - |z|^2 + k = 0\}$$

contains a bounded connected component, then it must contain a bounded connected component that surrounds the origin.

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Corollary (K. & Simanek, arXiv preprint)

Under the hypotheses of the key lemma, the set $\Gamma_{n,j}(C,k)$ *contains at most one bounded connected component surrounding the origin.*

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- Notice $f_{n,j,C,k}(0, \theta) > 0$.
- If $\Gamma_{n,j}(C, k)$ contained a bounded connected component, but it did not surround the origin, it would require the bounded connected component to surround a region where $f_{n,j,C,k}(0, \theta) < 0$.

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- A key observation in proving this lemma comes from supposing a contradictory statement.
- Notice $f_{n,j,C,k}(0,\theta) > 0$.
- If $\Gamma_{n,j}(C,k)$ contained a bounded connected component, but it did not surround the origin, it would require the bounded connected component to surround a region where $f_{n,j,C,k}(0,\theta) < 0$.
- $\Delta f_{n,j,C,k}(r,\theta) = -4$ $\Rightarrow\!\Leftarrow$

An Example

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Calculate the torsional rigidity for the unique bounded connected component determined by the set

$$\Gamma_{4,4}(C,k) = \{z : \mathrm{Re}[Cz^4 + z^4] - |z|^2 + k = 0\}$$

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$$\Gamma_{4,4}(C,k) = \{z : \mathrm{Re}[Cz^4 + z^4] - |z|^2 + k = 0\}$$

- We may let $\hat{C} = C + 1$ so that

$$\Gamma_{4,4}(C,k) = \{z : \mathrm{Re}[\hat{C}z^4] - |z|^2 + k = 0\}$$

and we satisfy the conditions of the monomial theorem.

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- Thus, $\Gamma_{4,4}(C, k)$ has a bounded connected component if and only if $\hat{C} \leq \frac{1}{4k}$.

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• A parameterization of this component is given in polar coordinates by

$$\{(\alpha,\theta), 0 \leq \theta \leq 2\pi\}$$

where

$$\alpha := \left(\frac{1 - \sqrt{1 - 4\hat{C}k\cos(4\theta)}}{2\hat{C}\cos(4\theta)} \right)^{1/2}.$$

An Example

- Recall,

$$\rho(\Omega) = \int_\Omega |\bar{z} - f(z)|^2 dA(z)$$

where Ω is the bounded connected component described by the above parameterization.

$$f(z) = \frac{d}{dz}\left[F(z)\right],$$

and

$$F(z) = \frac{\hat{C}z^4 + k}{2}.$$

An Example

- We have,

$\hat{C}$	$\rho(\Omega)$
$\frac{1}{4k}$	$1.63988k^2$
$\frac{1}{5k}$	$1.60815k^2$
$\frac{1}{10k}$	$1.57894k^2$
$\frac{1}{100k}$	$1.57087k^2$
$\vdots$	$\vdots$
0	$\frac{\pi}{2}k^2$

An Example

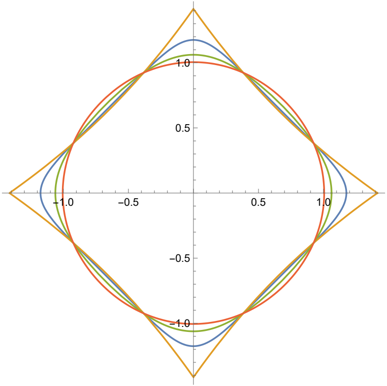

Figure: *A plot of the set* $\{(\alpha, \theta), 0 \leq \theta \leq 2\pi\}$ *with* $k = 1$ *for four different values of* $\hat{C}$*, ranging from* $\hat{C} = \frac{1}{4k}$ *at the outermost connected component in orange,* $\hat{C} = \frac{1}{5k}$ *in blue,* $\hat{C} = \frac{1}{10k}$ *in green, and* $\hat{C} = \frac{1}{100k}$ *as the nearly circular connected component in red.*

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