On compactness of products of Toeplitz operators

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Set-up

- $\Omega \subset \mathbb{C}^n$ is a **domain**.
  - **Polydisc** $\mathbb{D}^n = \mathbb{D} \times \cdots \times \mathbb{D}$
  - **Ball** $\mathbb{B}_n = \{z \in \mathbb{C}^n : \|z\| < 1\}$
- $b\Omega$ is the **boundary** of Ω.

  For $\Omega = \mathbb{D}^n$,

  - **Topological Boundary** $b\mathbb{D}^n$
  - **Distinguished Boundary** $\mathbb{T} \times \cdots \times \mathbb{T}$

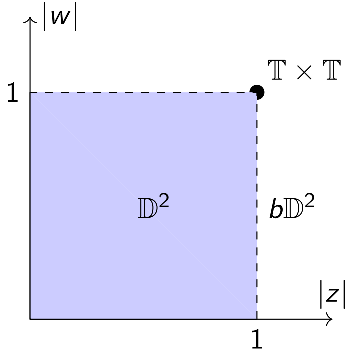

Figure: Boundary of a Bidisc

Bergman Space

- $L^2(\Omega) = \left\{ f : \Omega \to \mathbb{C} : \|f\|^2 = \int |f|^2 dV < \infty \right\},$
- **Bergman Space**

$$A^2(\Omega) = \{ f \in L^2(\Omega) : f \text{ is holomorphic} \}.$$

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$A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$, a Hilbert space. Thus, there exists an orthogonal projection, called **Bergman projection**

$$P : L^2(\Omega) \to A^2(\Omega).$$

Operators

For symbol $\phi \in L^\infty(\Omega)$,

Multiplication operator: $M_\phi : A^2(\Omega) \to L^2(\Omega)$

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Toeplitz operator: $T_\phi : A^2(\Omega) \to A^2(\Omega) \subset L^2(\Omega)$

$$T_\phi = PM_\phi.$$

Compactness

Definition 1

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Fact 1

Let H_1, H_2 be Hilbert spaces and $T : H_1 \to H_2$ be a linear map. Then T is **compact** *if and only if $Tf_n \to 0$ whenever $f_n \to 0$ weakly.*

Compactness

Problem 1

Characterize compactness of Toeplitz operators.

Axler-Zheng Theorem

- $A^2(\Omega)$ is a Reproducing Kernel Hilbert Space.
- The **Bergman kernel** $K_z = K(\cdot, z) \in A^2(\Omega)$ for $z \in \Omega$ is defined by

$$f(z) = \langle f, K_z \rangle \text{ for all } f \in A^2(\Omega).$$

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- The **normalized Bergman kernel**

$$k_z(w) = \frac{K(w, z)}{\sqrt{K(z, z)}}.$$

Note that $\|k_z\| = 1$.

The **Berezin transform** $\widetilde{T}$ of $T : A^2(\Omega) \to A^2(\Omega)$ at z is defined as

$$\widetilde{T}(z) = \langle Tk_z, k_z \rangle.$$

Theorem (Axler-Zheng 1998)

Let T be a finite sum of finite products of Toeplitz operators (with symbols in $L^\infty(\mathbb{D})$) on $A^2(\mathbb{D})$. Then

$$T \text{ is compact } \iff \widetilde{T} = 0 \text{ on } b\mathbb{D}.$$

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Question 1

Can we characterize compactness of T in terms of the boundary behavior of the symbols on the boundary?

Ball

Theorem (Coburn 1973)

There is a $*$-isomorphism $\sigma : \tau(\mathbb{B}_n)/\mathscr{K} \to C(b\mathbb{B}_n)$ satisfying

$$\sigma(T_f + \mathscr{K}) = f|_{b\mathbb{B}_n},$$

where

- *$\tau(\mathbb{B}_n)$ is the Toeplitz algebra generated by $\{T_\varphi : \varphi \in C(\overline{\mathbb{B}_n})\}$,*
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As a consequence, we see that for $f, g \in C(\overline{\mathbb{B}_n})$,

$$T_f T_g \text{ is compact } \Leftrightarrow fg = 0 \text{ on } b\mathbb{B}_n.$$

Polydisc

Motivated by Coburn's result, one may expect that the necessary and sufficient condition for $T_f T_g$ to be compact on $A^2(\mathbb{D}^n)$ is that fg vanishes on $b\mathbb{D}^n$.

Main Result

Theorem 1 (Le-R.-Şahutoğlu)

Let $f, g \in C(\overline{\mathbb{D}^2})$. *Then* $T_f T_g$ *is compact on* $A^2(\mathbb{D}^2)$ *if and only if*

$$T_{f(\xi,\cdot)} T_{g(\xi,\cdot)} = T_{f(\cdot,\xi)} T_{g(\cdot,\xi)} = 0.$$

on $A^2(\mathbb{D})$ *for all* $\xi \in \mathbb{T}$.

Main Ingredients of the Proof

Lemma 1

Let $q = (\xi, q_2) \in \mathbb{T} \times \overline{\mathbb{D}}$. *Then,*

$$\lim_{p \to q} \left\| T_{f - f(\xi, \cdot)} T_{g(\xi, \cdot)} k_p \right\| = \lim_{p \to q} \left\| T_f T_{g - g(\xi, \cdot)} k_p \right\| = 0.$$

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Theorem (Axler-Zheng Theorem for a Polydisc)

Let T *be a finite sum of finite products of Toeplitz operators (with symbols in* $L^\infty(\mathbb{D}^n)$*) on* $A^2(\mathbb{D}^n)$. *Then*

$$T \text{ is compact } \iff \widetilde{T} = 0 \text{ on } b\mathbb{D}^n.$$

Sketch of the Proof

($\Rightarrow$) Let $\xi \in \mathbb{T}$.

- Write $f = f - f(\xi,\cdot) + f(\xi,\cdot)$ and $g = g - g(\xi,\cdot) + g(\xi,\cdot)$. Then,

$$T_f T_g = T_{f(\xi,\cdot)} T_{g(\xi,\cdot)} + T_{f-f(\xi,\cdot)} T_{g(\xi,\cdot)} + T_f T_{g-g(\xi,\cdot)}.$$

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- By Lemma 1, for any $q = (\xi, q_2) \in \mathbb{T} \times \overline{\mathbb{D}}$,

$$\lim_{p \to q} \| T_f T_g k_p - T_{f(\xi,\cdot)} T_{g(\xi,\cdot)} k_p \| = 0,$$

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- Fix $p_2 \in \mathbb{D}$, then by the fact that $k_z^{\mathbb{D}^2}(w) = k_{z_1}^{\mathbb{D}}(w_1) \cdot k_{z_2}^{\mathbb{D}}(w_2)$,

$$\lim_{p_1 \to \xi} \| T_{f(\xi,\cdot)} T_{g(\xi,\cdot)} k_p \| = \lim_{p_1 \to \xi} \| k_{p_1}^{\mathbb{D}} T_{f(\xi,\cdot)} T_{g(\xi,\cdot)} k_{p_2}^{\mathbb{D}} \| = 0$$

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($\Leftarrow$) One can use Axler-Zheng Theorem for bidisc.

Applications

Corollary 1

Let $f, g \in C(\overline{\mathbb{D}^2})$. If $T_f T_g$ is compact on $A^2(\mathbb{D}^2)$, then $fg = 0$ on $\mathbb{T} \times \mathbb{T}$.

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- By Theorem 1,

$$T_{f(\xi,\cdot)} T_{g(\xi,\cdot)} = 0,$$

  on $A^2(\mathbb{D})$ for all $\xi \in \mathbb{T}$.

- By Coburn's result,

$$f(\xi,\cdot)g(\xi,\cdot) = 0 \text{ on } \mathbb{T}.$$

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- By Coburn's result,
$$f(\xi,\cdot)g(\xi,\cdot) = 0 \text{ on } \mathbb{T}.$$
- Thus, $fg = 0$ on $\mathbb{T} \times \mathbb{T}$.

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Let φ and ψ be two functions in $C(\overline{\mathbb{D}})$. We define $f(z,w) = \varphi(w)$ and $g(z,w) = \psi(w)$ for $z, w \in \overline{\mathbb{D}}$.

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By Theorem 1,

$$\begin{aligned} T_f T_g \text{ is compact on } A^2(\mathbb{D}^2) \Leftrightarrow & T_\varphi T_\psi = 0, \text{ and} \\ & \varphi(\xi)\psi(\xi) = 0 \text{ for all } \xi \in \mathbb{T}. \end{aligned}$$

Applications

Example 1

Let

$$\varphi(w) = \begin{cases} 1 - 2|w| & \text{for } 0 \leq |w| \leq \frac{1}{2} \\ 0 & \text{for } |w| > \frac{1}{2}, \end{cases}$$

and

$$\psi(w) = \begin{cases} 0 & \text{for } 0 \leq |w| \leq \frac{1}{2} \\ 2|w| - 1 & \text{for } |w| > \frac{1}{2}. \end{cases}$$

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- $T_\varphi T_\psi \not\equiv 0$ on $A^2(\mathbb{D})$, thus $T_f T_g$ is **NOT** compact on $A^2(\mathbb{D}^2)$.
- $\varphi\psi = 0$ on $\overline{\mathbb{D}}$. Then for $f(z,w) = \varphi(w)$ and $g(z,w) = \psi(w)$, we have $fg = 0$ on $\overline{\mathbb{D}^2}$

Applications

This example shows that the vanishing of fg on $b\mathbb{D}^2$ (or even on $\overline{\mathbb{D}^2}$) does not imply the compactness of $T_f T_g$.

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We see that $fg = 0$ on $b\mathbb{D}^2$ is not a sufficient condition for the compactness of $T_f T_g$. Is it a necessary condition?

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It turns out this question is related to the *zero-product problem* for Toeplitz operators on the disc.

Applications

For $\xi \in \mathbb{T}$ and $z, w \in \mathbb{D}$, we have

$$f(\xi, w)g(\xi, w) = \varphi(w)\psi(w)$$

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Proposition 1

Consider $T_f T_g$ *on* $A^2(\mathbb{D}^2)$ *such that* $f(z,w) = f_1(z)f_2(w)$ *and* $g(z,w) = g_1(z)g_2(w)$. *Then the following statements hold.*

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- *If* $T_f T_g$ *is a* **nonzero** *compact operator, then* $fg = 0$ *on* $b\mathbb{D}^2$.
- *If* $fg = 0$ *on* $b\mathbb{D}^2$ *and* fg *is* **not identically zero** *on* $\mathbb{D}^2$, *then* $T_f T_g$ *is compact.*

Q&A

Thank you!

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