Moments of Multivariate Trace Polynomials

George Roman

Joint work with Mike Jury

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Moments of Random Unitaries

Let $U(d)$ be the group of $d \times d$ unitary matrices, and consider the integral

$$\int_{U(d)} \prod_{j=1}^{k} \mathrm{tr}(U^j)^{a_j} \overline{\mathrm{tr}(U^j)^{b_j}} \, dU$$

where $a_j, b_j \in \mathbb{N}$. Integral with respect to Haar measure on $U(d)$.

Power Sum Symmetric Polynomials

Define d-variable polynomials

$$p_k(t_1, \ldots, t_d) := \sum_{i=1}^{d} t_i^k \quad , \quad k \in \mathbb{N}.$$

If $U \in U(d)$ has eigenvalues $x_1, \ldots, x_d$, we say

$$p_k(U) := p_k(x_1, \ldots, x_d) = \mathrm{tr}(U^k).$$

A product of such p_k polynomials is denoted

$$p_\lambda := \prod_{i=1}^{r} p_{\lambda_i} \quad , \quad \lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{N}^r.$$

Moments of Random Unitaries

Consequently,

$$\int_{U(d)} \prod_{j=1}^{k} \mathrm{tr}(U^j)^{a_j} \overline{\mathrm{tr}(U^j)^{b_j}} \, dU = \int_{U(d)} p_\lambda(U) \overline{p_\mu(U)} \, dU$$

where $\lambda = (\underbrace{1, \ldots, 1}_{a_1 \text{ times}}, \ldots, \underbrace{k, \ldots k}_{a_k \text{ times}})$ and similarly for μ with b_i's.

Moments of Random Unitaries

Diaconis and Shahshahani (1994) use representation theory and combinatorics to show

$$\int_{U(d)} p_\lambda(U)\overline{p_\mu(U)}\, dU = \prod_{j=1}^{k} \delta_{a_j,b_j} j^{a_j} a_j! =: \delta_{\lambda,\mu} z_\lambda$$

for any $d \geq 1a_1 + 2a_2 + \cdots + ka_k$.

More Matrix Variables

What happens if we consider multiple independent unitary variables $U_1, \ldots, U_g$. For example

$$\lim_{d \to \infty} \int_{U(d)^3} \mathrm{tr}(U_1 U_2)\mathrm{tr}(U_1 U_3)\overline{\mathrm{tr}(U_1 U_1)\mathrm{tr}(U_3)\mathrm{tr}(U_2)}\, dU^3 = ???$$

A couple problems:

- How do we keep track of the U_i's?
- How do we determine where the traces go?

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Words

Given a set of distinct symbols (letters) $\{a, b, c, d\}$, a word is a finite string of these letters e.g. *aabcddc* or *bcdbd*. Set of words of length n is W_n. Symmetric group S_n acts on W_n by permuting letters:

Example

$$(123)(45) * abacd = aabdc$$

Partitions

If $(\lambda_1, \ldots, \lambda_r) \in \mathbb{N}_+^r$ where $\sum \lambda_i = n$, call λ a partition of n. Partitions of n in bijection with cycle types of permutations in S_n.

Example

$$(2845)(16)(97)(3) \in S_9 \text{ corresponds to } \lambda = (4, 2, 2, 1).$$

Deeper connection: If σ has cycle type λ, then

$$\begin{aligned} z_\lambda &= \left|\{\tau \in S_n \mid \tau\sigma = \sigma\tau\}\right| \\ &= \left|\{\tau \in S_n \mid \tau\sigma\tau^{-1} = \sigma\}\right|. \end{aligned}$$

i.e. $z_\lambda =$ size of stabilizer for conjugation action $\tau * \sigma = \tau\sigma\tau^{-1}$.

Condensing the Trace Products

Theorem (Kostant, 1958)

If $\sigma = (i_1\, i_2\, \cdots\, i_s) \cdots (j_1\, j_2\, \cdots\, j_t) \in S_n$ and $X_1, \ldots, X_n \in M(d)$, then

$$\mathrm{tr}(X_{i_1} X_{i_2} \cdots X_{i_s}) \cdots \mathrm{tr}(X_{j_1} X_{j_2} \cdots X_{j_t}) = \mathrm{tr}(\sigma^{-1} \circ X_1 \otimes \cdots \otimes X_n).$$

If σ has cycle type λ and we choose all $X_i = U$, reduces to

$$p_\lambda(U) = \mathrm{tr}(\sigma^{-1} \circ U^{\otimes n}).$$

Note σ, σ' have same cycle type if and only if $\sigma = \tau\sigma'\tau^{-1}$ for some $\tau \in S_n$.

Generalized Trace Polynomials

Definition

For $U_1, \ldots, U_g \in U(d)$, $\sigma \in S_n$, and $w \in W_n$, define

$$p_{\sigma,w}(U) := \mathrm{tr}(\sigma^{-1} \circ U^{\otimes w})$$

Follows from definitions that $p_{\sigma,w} = p_{\tau\sigma\tau^{-1},\tau(w)}$ for all $\tau \in S_n$. If Λ is an orbit of $S_n \times W_n$ under S_n-action, we say

$$p_\Lambda := p_{\sigma,w}$$

for any $(\sigma, w) \in \Lambda$.

Integrating Trace Polynomials

Theorem (Jury, R., 2024+)

If Λ, M are orbits of $S_n \times W_n$, then

$$\lim_{d \to \infty} \int_{U(d)^g} p_\Lambda(U) \overline{p_M(U)} \, dU^g = \delta_{\Lambda, M} |\mathrm{stab}(\sigma, w)|$$

where $(\sigma, w) \in \Lambda$.

Compare to known case when $g = 1$

$$\int_{U(d)} p_\lambda(U) \overline{p_\mu(U)} \, dU = \delta_{\lambda, \mu} |\mathrm{stab}(\sigma)|.$$

Non-Stability

If $g = 1$, Diaconis and Shahshahani show integral is constant once d is large enough. Not true for $g > 1$.

Example

$$\int_{U(d)^2} \mathrm{tr}(U_1 U_1 U_2 U_2)\overline{\mathrm{tr}(U_1 U_1 U_2 U_2)}\, dU^2 = \frac{d^2}{d^2 - 1}$$

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Another important kind of integral over the unitary group is

$$\int_{U(d)} \det(1 - zU)\det(1 - \overline{w}U^*)\, dU = 1 + z\overline{w} + (z\overline{w})^2 + \cdots + (z\overline{w})^d.$$

Observation

If $|z\overline{w}| < 1$, then

$$\lim_{d\to\infty} \int_{U(d)} \det(1 - zU)\det(1 - \overline{w}U^*)\, dU = \frac{1}{1 - z\overline{w}}.$$

Multiple Unitaries

Theorem (Jury, R., 2024+)

If $z, w \in \mathbb{C}^k$ with $|z|, |w| < 1$, then

$$\lim_{d\to\infty} \int_{U(d)^g} \det\left(1 - \sum_{i=1}^{g} z_i U_i\right) \det\left(1 - \sum_{i=1}^{g} \overline{w_i} U_i^*\right) \, dU^g = \frac{1}{1 - \langle z, w\rangle}.$$

Proof outline:

- Expand determinant as sum of products of minor determinants of U_i's.
- Integral of product = product of integrals (by independence).
- Substitute $z_i = w_i = 0$ for $i = 2, \ldots, k$ to reduce to already known 1-variable case to find value of integrals.

Matrix Coefficients with Multiple Unitaries

Conjecture

If $||\Sigma_i\, X_i \otimes \overline{Y_i}|| < 1$, then

$$\int_{U(d)^g} \det(1 - \Sigma_i\, X_i \otimes U_i) \det(1 - \Sigma_i\, Y_i^* \otimes U_i^*)\, dU^g \longrightarrow \det(1 - \Sigma_i\, X_i \otimes \overline{Y_i})^{-1}$$

as $d \to \infty$.

- True by Szegő limit theorem if $g = 1$ and X_1, Y_1 matrices.
- Proven true if $g > 1$ and X_i, Y_i scalars.
- Formal proof uses series expansion of determinant in terms of p_Λ, challenge justifying limits.

Future Questions

- Diaconis and Shahshahani examine integrals over orthogonal and symplectic matrices too. What is the multivariable behavior for these groups?
- The p_Λ generalize power sum polynomials. Other canonical symmetric polynomials also exist, can we extend them to $S_n \times W_n$ setting?