Exposed points of matrix convex sets

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Noncommutative convexity – matrix convex sets

Definition: Let $K_n \subseteq M_n(V)$ for $n \in \mathbb{N}$ and denote $\mathbf{K} = (K_n)_{n\in\mathbb{N}}$.

1. A **matrix convex combination** of $A_1, \ldots, A_k \in \mathbf{K}$ with $A_i \in K_{n_i}$ is an expression of the form

$$\sum_{i=1}^{k} \gamma_i^* A_i \gamma_i \in M_n(V),$$

   where $\gamma_i \in \mathbb{M}_{n_i,n}$ satisfy $\sum_{i=1}^{k} \gamma_i^* \gamma_i = \mathbb{I}_n$.

2. The family $\mathbf{K}$ is a **matrix convex set** if it is closed under formation of matrix convex combinations of its elements.

Morphisms – matrix affine maps

Definition: A **matrix affine map** $\Phi = (\Phi_r)_r$ between matrix convex sets $\mathbf{K}$ and $\mathbf{L}$ in the spaces V and W, respectively, is a sequence of linear maps $\Phi_r : M_r(V) \to M_r(W)$ that satisfy $\Phi_r(K_r) \subseteq L_r$ for all $r \in \mathbb{N}$ and

$$\Phi_r\Big(\sum_{i=1}^{k} \gamma_i^* A_i \gamma_i\Big) = \sum_{i=1}^{k} (\gamma_i^* \otimes \mathbb{I}_r)\, \Phi_{r_i}(A_i)(\gamma_i \otimes \mathbb{I}_r)$$

for all matrix convex combinations $\sum_{i=1}^{k} \gamma_i^* A_i \gamma_i$.

Motivation

Matrix extreme points

Definition: Let $\mathbf{K} = (K_n)_{n\in\mathbb{N}}$ be a matrix convex set and $A \in K_n$.

1. A matrix convex combination

$$A = \sum_{i=1}^{k} \gamma_i^* A_i \gamma_i \tag{1}$$

   is **proper** if all the γ_i are surjective. In particular, $n \geq n_i$ for all i.

2. The point A is **matrix extreme** if any expression of the form (1) implies all the A_i are unitarily equivalent to A. Hence, $n_i = n$ for all i.

- Notation: $\text{mext}(\mathbf{K})$.

Well-known results

The Webster-Winkler matricial Krein-Milman theorem

Definition: Let $\mathbf{S} = (S_n)_{n\in\mathbb{N}}$ with $S_n \subseteq M_n(V)$. The smallest closed matrix convex set containing $\mathbf{S}$ is called the **closed matrix convex hull** of $\mathbf{S}$ and is denoted by $\overline{\text{mconv}}\,\mathbf{S}$.

Theorem (Webster-Winkler 99')

Let $\mathbf{K}$ be a compact matrix convex set in a locally convex space V. Then $\text{mext}\,\mathbf{K} \neq \emptyset$ and

$$\mathbf{K} = \overline{\text{mconv}}(\text{mext}\,\mathbf{K}).$$

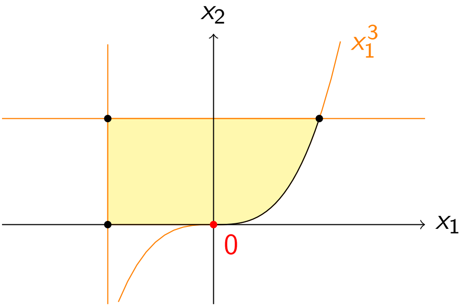
Exposed points

The Straszewicz theorem

Theorem (Straszewicz 35', Klee 58')

For any compact convex set K in a normed space V, $\exp(K) \neq \emptyset$ and

$$K = \overline{\text{conv}}(\exp(K)).$$



Matrix exposed points

Definition and basic properties

Definition: Let $\mathbf{K} = (K_n)_{n\in\mathbb{N}}$ be a matrix convex set in a dual vector space V. An element $A \in K_n$ is called a **matrix exposed point** of $\mathbf{K}$ if there exist a continuous linear map $\Phi : V \to \mathbb{M}_n$ and a self-adjoint matrix $\alpha \in \mathbb{M}_n$ such that the following conditions hold:

(a) for all positive integers r and $B \in K_r$ we have $\Phi_r(B) \preceq \alpha \otimes \mathbb{I}_r$;

(b) $\{B \in K_n \mid \alpha \otimes \mathbb{I}_n - \Phi_n(B) \succeq 0 \text{ singular}\} = \{U^*AU \mid U \in \mathbb{M}_n \text{ unitary}\}$.

Properties:

- The matrix exposed points in K_1 coincide with the ordinary exposed points of K_1.
- For $r < n$ and $B \in K_r$ the strict inequality $\Phi_r(B) \prec \alpha \otimes \mathbb{I}_r$ holds.

Matrix exposed points

Connection with matrix extreme points

Proposition (Kriel 19, Klep-Š)

Let $\mathbf{K} = (K_n)_{n \in \mathbb{N}}$ be a matrix convex set. Then:

(a) Every matrix exposed point in K_n is ordinary exposed in K_n.

(b) Any matrix exposed point is matrix extreme.

(c) A point which is both exposed and matrix extreme, is a matrix exposed point.

Matrix exposed points

Connection with matrix extreme points (a)

(a) Every matrix exposed point in K_n is ordinary exposed in K_n.

Proof idea: form the compression functional

$$\phi(X) = v^*(\alpha \otimes \mathbb{I}_n - \Phi_n(X))v$$

for some $v \in \mathbb{C}^n \otimes \mathbb{C}^n$. But which v?

Proposition (McCullough, Farenick)

Let $A \in K_n$ be a matrix exposed point with an exposing pair (Φ, α). Then the following statements hold.

(a) For any nonzero $x = \sum_{j=1}^n x_j \otimes e_j \in \mathbb{C}^n \otimes \mathbb{C}^n$ in the kernel of $\alpha \otimes \mathbb{I}_n - \Phi_n(A)$, the components $x_1, \ldots, x_n$ form a basis of $\mathbb{C}^n$.

(b) The kernel of $\alpha \otimes \mathbb{I}_n - \Phi_n(A)$ is one-dimensional.

Matrix exposed points

Connection with matrix extreme points (c)

(c) A point which is both exposed and matrix extreme, is also matrix exposed.

Proof idea:

- A exposed $\implies$ $\exists\varphi : M_n(V) \to \mathbb{R}$ and $a \in \mathbb{R}$ with $\varphi(A) = a$ and $\varphi|_{K_n\setminus\{A\}} < a$.
- Idea: use Effros-Winkler matricial separation techniques, but the separation happens over a real closed field (cf. Netzer-Thom separation theorem).
- A finiteness theorem gives domination of φ by a state $p : \mathbb{M}_n \to \mathbb{C}$.
- Proceed by a GNS-type construction as in the proof of the matricial Hahn-Banach theorem.
- A matrix extreme $\implies$ $A \notin \mathrm{mconv}(K_n\setminus\{U^*AU \mid U \in \mathbb{U}_n\})$. Used in the end to prove the separation properties of the obtained map.

Matrix exposed points

The matricial Straszewicz theorem

Theorem (Matrix Straszewicz, Kriel, Klep-Š)

Let $\mathbf{K}$ be a compact matrix convex set in a normed vector space V. Then $\operatorname{mexp}\mathbf{K} \neq \emptyset$ and

$$\mathbf{K} = \overline{\operatorname{mconv}}(\operatorname{mexp}\mathbf{K}).$$

Proof uses the Hartz-Lupini method of introducing an associated family of convex sets $\{\Gamma_n(\mathbf{K})\}_{n\in\mathbb{N}}$ to a matrix convex set $\mathbf{K} = (K_r)_{r\in\mathbb{N}}$ given by

$$\Gamma_n(\mathbf{K}) = \{(\gamma^*\gamma, \gamma^*A\gamma) \mid \gamma \in \mathbb{M}_{k,n}, \operatorname{tr}(\gamma^*\gamma) = 1, k \in \mathbb{N}, A \in K_k\}.$$

Matrix exposed points

The matricial Straszewicz theorem – idea of the proof

Proposition (Klep-Š)

Let $\mathbf{K} = (K_m)_{m\in\mathbb{N}}$ be a matrix convex set and $A \in K_r$. Let $\gamma \in \mathbb{M}_{r,n}$ be a surjective matrix with $\mathrm{tr}(\gamma^*\gamma) = 1$ such that the point $(\gamma^*\gamma, \gamma^* A\gamma)$ is exposed in $\Gamma_n(\mathbf{K})$. Then A is a matrix exposed point of $\mathbf{K}$.

Proof idea – as in the WW proof of the matricial Krein-Milman theorem:

use Hahn-Banch separation + the above Proposition to reduce to the classical Straszewicz-Klee theorem.

More is true - a density result

Proposition (Klep-Š)

Let $\mathbf{K}$ be a compact matrix convex set in a normed vector space V. Then the matrix exposed points of $\mathbf{K}$ are dense in the matrix extreme points of $\mathbf{K}$.

Proof is again reduction to the classical case using the sets $\Gamma_n(\mathbf{K})$.

Corollary

If all the extreme points of a matrix convex set $\mathbf{K}$ are exposed, then all the matrix extreme points of $\mathbf{K}$ are matrix exposed.

- This is, e.g., the case with free spetrahedra.
- But also with the **matrix state space** of any separable unital C^*-algebra $\mathcal{A}$. This is the family $\mathbf{UCP}(\mathcal{A}) = (\mathbf{UCP}_n(\mathcal{A}))_n$, where

$$\mathbf{UCP}_n(\mathcal{A}) = \{\Phi : \mathcal{A} \to \mathbb{M}_n \mid \Phi \textit{ unital completely positive}\}.$$

It is a weak* compact matrix convex set in $\mathcal{A}^*$.

Examples

Matrix state spaces of separable unital C^*-algebras continued

- There is a linear bijection between $\mathbf{UCP}_n(\mathcal{A})$ and the state space of $M_n(\mathcal{A})$ sending any $\Phi : \mathcal{A} \to \mathbb{M}_n$ to

$$\tilde{\Phi} : M_n(\mathcal{A}) \to \mathbb{C},$$
$$\tilde{\Phi}(X) = \frac{1}{n}\langle \Phi_n(X)e, e\rangle,$$

  where $e = e_1 \oplus \cdots \oplus e_n$ and $\{e_i\}_i$ is the standard basis of $\mathbb{C}^n$.

- In the state space of a separable unital C^*-algebra, extreme points are exposed (Alfsen).

Thank you!