Denjoy-Wolff points on the bidisc

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Joint work with Michael Jury

Fixed points on the disc I

Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic, not conjugate to a rotation.
Set $f^{[0]} = I$, $f^{[n+1]} = f \circ f^{[n]}$ and write $\mathbb{T} = \partial\mathbb{D}$.

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First, assume f has a fixed point $z_0 \in \mathbb{D}$.
Then, z_0 is unique and

$$f^{[n]} \to z_0 \quad \text{locally uniformly.}$$

In this setting, z_0 will be called the Denjoy-Wolff point of f.

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Theorem (Denjoy-Wolff, 1926)

In this setting,

$$f^{[n]} \to \tau \quad \text{locally uniformly.}$$

Boundary regularity on the bidisc

Definition

Let $f : \mathbb{D}^2 \to \mathbb{D}$ be holom. $\tau \in \partial\mathbb{D}^2$ will be called a carapoint for f if

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Theorem (Agler-McCarthy-Young, 2012)

Let $f : \mathbb{D}^2 \to \mathbb{D}$ be holom. with $\tau \in \partial\mathbb{D}^2$ a carapoint. Then,

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(III) The function $D_{(.)}f(\tau)$ can be described using certain one-variable *Pick class* functions (see also Agler–Tully-Doyle–Young, 2012).

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- $D_\delta f(1,1) = \lim_{t \to 0+} \dfrac{f\big((1,1) + t\delta\big) - 1}{t} = \dfrac{2\delta_1\delta_2}{\delta_1 + \delta_2}.$

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$D_\delta f(1,1)$ not linear in δ, hence no "nt gradient" at $(1,1)$.

Type I functions

Let $f : \mathbb{D}^2 \to \mathbb{D}$ be holomorphic, $f(z,w) \not\equiv z$. For $w \in \mathbb{D}$, define the (left) slice function $f_w : \mathbb{D} \to \mathbb{D}$ by

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Example

Let

$$f(z,w) = \frac{1 - zw}{2 - z - w}.$$

Every f_w satisfies $f_w(1) = 1$ and has a (nt) derivative equal to 1 at 1. Thus, 1 is the common Denjoy-Wolff point of all slices f_w.

Type II functions

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Since $f_w(w) = w$, the Denjoy-Wolff point of f_w is the interior point $\xi(w) = w$, for all $w \in \mathbb{D}$.

A critical dichotomy

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So, any "reasonable" definition of Denjoy-Wolff points for $\mathbb{D}^2$ should respect this dichotomy.

The componentwise radial derivative

Let $\tau = (\tau_1, \tau_2) \in \mathbb{T}^2$ be a carapoint for $f : \mathbb{D}^2 \to \mathbb{D}$. For $M > 0$, consider the "componentwise radial" directional derivative

$$D_{(\tau_1, M\tau_2)} f(\tau) = \lim_{t \to 0+} \frac{f\big((1-t)\tau_1, (1-Mt)\tau_2\big) - f(\tau)}{-t}.$$

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For $M = 1$, one can show

$$\frac{D_{(\tau_1, \tau_2)} f(\tau)}{f(\tau)} = \lim_{r \to 1^-} \frac{1 - |f(r\tau)|}{1 - |r|}.$$

DW points on the bidisc

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- a (left) Type I DW point if $K_\tau(M) \leq 1$ for all $M > 0$;
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Theorem (Jury-T., 2023)

Having a Type I DW point $\Leftrightarrow$ being a Type I function,
Having a Type II DW point $\Rightarrow$ being a Type II function (no converse!)

Hervé's results

Let $F = (f, g) : \mathbb{D}^2 \to \mathbb{D}^2$ holomorphic and fixed-point-free.
Put $F^{[n]} = F \circ F \circ \cdots \circ F$. $\{F^{[n]}\}$ may not converge!

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Hervé (1954) studied

$$L_F := \left\{ G : \mathbb{D}^2 \to \overline{\mathbb{D}}^2 \;\middle|\; \exists \{n_k\} \text{ s.t. } F^{[n_k]} \to G \text{ locally uniformly} \right\}$$

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by looking at three cases:

- $(f, g) =$ (Type I, Type II);
- $(f, g) =$ (Type I, Type I);
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New Perspective

Having DW points where f or g are not "too regular" leads to stronger convergence results!

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In this setting:

Theorem (Hervé, 1954)

Either **(P1)** : $L_F \subset \left\{(\tau_1, \psi) \mid \psi : \mathbb{D}^2 \to \overline{\mathbb{D}} \text{ analytic}\right\}$

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If f doesn't have a nt gradient at τ, then **(P1)**.

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If g doesn't have a nt gradient at τ, then **(P2)**.

A (Type I, Type I) example

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Example

Let $F = (f, g) : \mathbb{D}^2 \to \mathbb{D}^2$, where

$$f(z,w) = \frac{3zw - z - w - 1}{zw + z + w - 3}, \qquad g(z,w) = \frac{1 - zw}{2 - z - w}.$$

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Then, $(1,1)$ is a Type I DW point for both f and g.
Refinement (Jury, T.): neither f nor g has a nt gradient at $(1,1)$.
Thus,

$$F^{[n]} \to (1,1) \quad \text{locally uniformly.}$$

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$L_F \subset \left\{(\tau_1, \psi) \mid \psi : \mathbb{D}^2 \to \overline{\mathbb{D}} \text{ analytic}\right\}$

(Type I, Type II)

Assume $(f, g) = $ (Type I, Type II). Then, $\exists \tau_1 \in \mathbb{T}$ such that

$$\{\tau_1\} \times \overline{\mathbb{D}} \text{ consists of Type I DW point for } f.$$

However, g needn't have any Type II DW points.
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If, in addition, there exists $\tau_2 \in \mathbb{T}$ such that (τ_1, τ_2) is a Type II DW point for g and f does not have a nt gradient at (τ_1, τ_2), then

$$F^{[n]} \to (\tau_1, \tau_2) \quad \text{locally uniformly.}$$

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Example

Let $F = (f, g) : \mathbb{D}^2 \to \mathbb{D}^2$, where

$$f(z, w) = \frac{1 - zw}{2 - z - w}, \qquad g(z, w) = \frac{z + w - 2zw}{2 - z - w}.$$

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Refinement (Jury, T.): f does not have a nt gradient at $(1, 1)$. Thus,

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In this setting:

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$F^{[n]} \to \tau$ locally uniformly.

Key tool: Agler's representation

Given $f : \mathbb{D}^2 \to \mathbb{D}$ holomorphic, there exists a Hilbert space $M = M^1 \oplus M^2$ and a (holomorphic) map

$$u_{(\cdot)} = (u^1_{(\cdot)}, u^2_{(\cdot)}) : \mathbb{D}^2 \to M^1 \oplus M^2$$

such that for all $z = (z_1, z_2), w = (w_1, w_2) \in \mathbb{D}^2$

$$1 - f(z)\overline{f(w)} = (1 - z_1\overline{w_1})\langle u^1_z, u^1_w \rangle + (1 - z_2\overline{w_2})\langle u^2_z, u^2_w \rangle$$

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$$\frac{D_\delta f(\tau)}{f(\tau)} = \overline{\tau_1}\delta_1 ||x^1_\tau(\delta)||^2 + \overline{\tau_2}\delta_2 ||x^2_\tau(\delta)||^2.$$