Fredholm and frame-preserving weighted composition operators

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Introduction

In this talk we characterize Fredholm and frame-preserving weighted composition operators on some general Hilbert spaces of holomorphic functions in bounded domains in $\mathbb{C}^n$. Our work generalizes some recent results by Cao, He and Zhu [Bull. London Math Soc., 2019] on Fredholm composition operators to weighted composition operators, and also generalizes some of our own recent results on frame-preserving weighted composition operators [Manhas, Prajitura, Z, IEOT, 2019] from general weighted Bergman Hilbert spaces to some more general Hilbert spaces.

Notations

• Ω: a bounded domain in $\mathbb{C}^n$; $\partial\Omega$: the boundary of Ω.

• $\mathcal{H}(\Omega)$: the space of all holomorphic functions on Ω.

• $\mathrm{Aut}(\Omega)$: the automorphism group of Ω. Hence $\varphi \in \mathrm{Aut}(\Omega)$ if and only if $\varphi : \Omega \to \Omega$ is holomorphic, one-to-one, and onto.

• **Weighted composition operators.** For a holomorphic function ψ and a holomorphic self-map φ on Ω, the weighted composition operator $W_{\psi,\varphi}$ on $\mathcal{H}(\Omega)$ is defined as

$$W_{\psi,\varphi} f = M_\psi C_\varphi f = \psi \cdot (f \circ \varphi), \qquad f \in \mathcal{H}(\Omega),$$

where M_ψ is the (pointwise) multiplication operator with symbol ψ and C_φ is the composition operator with symbol φ.

Fredholm operators

We denote by H be a complex Hilbert space of functions in $\mathcal{H}(\Omega)$. Let $T : H \to H$ be a bounded linear function. T is called a Fredholm operator if T has closed range, $\dim(\ker(T)) < \infty$, and $\dim(\ker(T^*)) < \infty$. Obviously, if T is invertible on H then it is Fredholm.

Standing assumptions for Ω and H

Following Cao, He and Zhu in [BLMS 2019], we make the same assumptions (a)-(c) as below for Ω and H throughout the paper.

(a) For each $w \in \Omega$, the evaluation map $f \to f(w)$ is a bounded linear functional on H.

(b) Every $\varphi \in \mathrm{Aut}(\Omega)$ induces a bounded composition operator C_φ on H.

(c) The points of Ω are separated by H in the sense that if $\{a_k\}$ is any finite sequence of distinct points in Ω then the kernel functions $\{K_{a_k}\}$ are linearly independent.

Easy consequences of our standing assumptions

There are a couple of obvious consequences of the above assumptions.

1. From condition (a) we know from the Riesz representation theorem that there exists a unique function $K_w \in H$ such that

$$f(w) = \langle f, K_w \rangle, \quad f \in H.$$

The function $K(z, w) = K_w(z)$, defined on $\Omega \times \Omega$, is called the reproducing kernel of H.

2. From condition (b) we know that, for $\varphi \in \mathrm{Aut}(\Omega)$, the composition operator C_φ is invertible, and $C_\varphi^{-1} = C_{\varphi^{-1}}$.

The fourth assumption

(d) Given any sequence $\{z_k\}$ of points in Ω such that $\mathrm{dist}(z_k, \partial\Omega) \to 0$ as $k \to \infty$. Let

$$f_k(z) = K(z, z_k)/\sqrt{K(z_k, z_k)} = K(z, z_k)/\|K_{z_k}\|.$$

Then the sequence $\{f_k\}$ converges to 0 weakly in H.

We will use condition (d) to replace the following condition used in the Main Theorem in [Cao, He, Zhu, BLMS 2019]:

$$K(z, z) \to \infty \quad \text{as} \quad z \to \partial\Omega. \tag{1}$$

In the proof of the Main Theorem of the above paper, it assumes that condition (1) implies condition (d). However, the next example shows that condition (d) actually cannot be obtained from condition (1). We thank Kehe Zhu for pointing out this fact and for providing the following example to us.

An example

Let $\Omega = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and let H denote the Hilbert space of holomorphic functions on Ω with orthonormal basis $e_n = z^n$, $n = -1, 0, 1, 2, 3, \cdots$. The reproducing kernel of H is

$$K(z, w) = \frac{1}{z\bar{w}} + \frac{1}{1 - z\bar{w}}.$$

Obviously, $K(z, z) \to \infty$ as $z \to \partial\Omega$. Let $\{z_k\}$ be a sequence of points in Ω such that $|z_k| \to 0$ (hence $z_k \to \partial\Omega$), and let

$$f_k(z) = \frac{K(z, z_k)}{\sqrt{K(z_k, z_k)}} = \frac{|z_k|\sqrt{1 - |z_k|^2}}{z\bar{z}_k(1 - z\bar{z}_k)}.$$

Obviously, $f_k(z) \not\to 0$ pointwise as $k \to \infty$. Thus $\{f_k\}$ does not converge to 0 weakly in H.

Frames

We will also consider weighted composition operators that preserve frames. Recall that a *frame* for a Hilbert space H with an inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|_H$ is a family of vectors $f_i \in H$, $i \in \mathbb{N}$, for which there are constants $A, B > 0$ satisfying:

$$A\|f\|_H^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \leq B\|f\|_H^2, \qquad \text{for all } f \in H. \tag{2}$$

The concept of frames was first introduced by Duffin and Schaeffer in the context of nonharmonic Fourier series in [TAMS, 1952]. It is the generalization of the concept of orthonormal sets in Hilbert spaces. The frame theory has broad applications in pure mathematics, applied mathematics, computer science, and engineering.
We say that a linear operator T on a Hilbert space H preserves frames if $\{Tf_i\}$ is a frame in H for any frame $\{f_i\}$ in H.

Let $\mathcal{M}(H)$ denote the space of all multipliers of H, that is

$$\mathcal{M}(H) = \{\psi \in \mathcal{H}(\Omega) : \ \psi f \in H \text{ for all } f \in H\}.$$

By the Closed Graph Theorem, we know that $\psi \in \mathcal{M}(H)$ is equivalent to say the multiplication operator M_ψ is a bounded operator on H.

Theorem 1

Let Ω be a bounded domain in $\mathbb{C}^n$, and let H be a Hilbert space of functions in $\mathcal{H}(\Omega)$ satisfying conditions (a)-(d). If $W_{\psi,\varphi} : H \to H$ is Fredholm, then we must have the following conditions.

(i) $\varphi \in \mathrm{Aut}(\Omega)$.

(ii) $\psi \in \mathcal{M}(H)$.

(iii) ψ *is bounded away from 0 near the boundary* $\partial\Omega$.

Fredholm weighted composition operators, $n > 1$

In Theorem 1 we obtained necessary conditions for a weighted composition operator to be Fredholm. Next let us investigate the inverse direction. It turns out that the situation is different for the case $n > 1$ and the case $n = 1$. We first consider the case $n > 1$. We need the following extra assumption for H and Ω.

(e) If $\psi \in \mathcal{M}(H)$ and $|\psi(z)| \geq \delta$ for some positive constant δ and all $z \in \Omega$, then $1/\psi \in \mathcal{M}(H)$.

We would like to point out that condition (e) is non-trivial. It is referred as "one-function corona theorem" for $\mathcal{M}(H)$ in [Fang, Xia, PAMS 2013], and it has been proved to be satisfied by the Drury-Arveson space in [Fang, Xia, PAMS 2013] and [Richter, Sunkes, PAMS 2016], and more generally, by Hardy-Sobolev spaces in [Cao, He, Zhu, JFA 2016] in the unit ball in $\mathbb{C}^n$.

Here is our main theorem for $n > 1$.

Theorem 2

Let $n > 1$. Let Ω be a bounded domain in $\mathbb{C}^n$, and let H be a Hilbert space of functions in $\mathcal{H}(\Omega)$ satisfying conditions (a)-(e). Then the following conditions are equivalent.

(i) $W_{\psi,\varphi}$ *is Fredholm.*

(ii) $W_{\psi,\varphi}$ *is invertible.*

(iii) $\varphi \in \mathrm{Aut}(\Omega)$, $\psi \in \mathcal{M}(H)$, *and* ψ *is bounded away from 0 near the boundary* $\partial\Omega$.

In [Manhas, Prajitura, Z, IEOT 2019], it is proved that, for the general weighted Bergman space A^2_α with $\alpha > -(n+1)$, a bounded weighted composition operator $W_{\psi,\varphi}$ is invertible if and only if it preserves frames. It is natural to ask whether the same result is true for the general reproducing kernel Hilbert spaces H of holomorphic functions on a bounded domain Ω. The following result answers this question for $n > 1$.

Corollary 3

Let $n > 1$. Suppose Ω is a bounded domain in $\mathbb{C}^n$, and suppose H is a Hilbert space of functions in $\mathcal{H}(\Omega)$ satisfying conditions (a)-(e). Suppose $W_{\psi,\varphi} : H \to H$ is bounded. Then the following conditions are equivalent.

(i) $W_{\psi,\varphi}$ *is Fredholm.*

(ii) $W_{\psi,\varphi}$ *is invertible.*

(iii) $W^*_{\psi,\varphi}$ *is bounded below.*

(iv) $W_{\psi,\varphi}$ *is surjective.*

(v) $W_{\psi,\varphi}$ *preserves frames of* H*.*

(vi) $\varphi \in \mathrm{Aut}(\Omega)$ *and* ψ *is bounded away from 0 near the boundary* $\partial\Omega$*.*

Fredholm weighted composition operators, $n = 1$

For $n = 1$, we need another extra assumption on H and Ω.

(f) If $f \in H$ and $f(a) = 0$ for some $a \in \Omega$, then $f/(z - a) \in H$.

Below is our main theorem for the case $n = 1$.

Theorem 4

Suppose Ω is a bounded domain in $\mathbb{C}$, and suppose H is a Hilbert space of functions in $\mathcal{H}(\Omega)$ satisfying conditions (a)-(f). Then the following conditions are equivalent.

(i) *$W_{\psi,\varphi}$ is Fredholm.*

(ii) *$\varphi \in \mathrm{Aut}(\Omega)$, $\psi \in \mathcal{M}(H)$, and ψ is bounded away from 0 near the boundary $\partial\Omega$.*

Comparing with Corollary 3, we have the following result on invertible and frame-preserving weighted composition operators for the case $n = 1$. Note that for this result we do not require condition (f).

Proposition 5

Suppose Ω is a bounded domain in $\mathbb{C}$, and suppose H is a Hilbert space of functions in $\mathcal{H}(\Omega)$ satisfying conditions (a)-(e). Suppose $W_{\psi,\varphi} : H \to H$ is bounded. Then the following conditions are equivalent.

(i) $W_{\psi,\varphi}$ *is invertible.*

(ii) $W_{\psi,\varphi}^*$ *is bounded below.*

(iii) $W_{\psi,\varphi}$ *is surjective.*

(iv) $W_{\psi,\varphi}$ *preserves frames of H.*

(v) $\varphi \in \mathrm{Aut}(\Omega)$ *and ψ is bounded away from 0 on Ω.*

THANKS!