# Non-commutative Ball Maps
## wip

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# Ball maps

- A *ball map* is a proper map $\varphi : \mathbb{B}^{\mathtt{g}} \to \mathbb{B}^{\mathtt{G}}$.
- e.g., $\varphi(z_1, z_2) = (z_1^2, \sqrt{2} z_1 z_2, z_2^2)$.
- Two ball maps $\varphi, \psi$ are *equivalent* if there exists automorphisms $f : \mathbb{B}^{\mathtt{g}} \to \mathbb{B}^{\mathtt{g}}$ and $g : \mathbb{B}^{\mathtt{G}} \to \mathbb{B}^{\mathtt{G}}$ such that
$$\varphi \circ f = g \circ \psi.$$
- A ball map $\varphi : \mathbb{B}^{\mathtt{g}} \to \mathbb{B}^{\mathtt{G}}$ is *minimal* if its image lies in no proper affine linear subspace of $\mathbb{C}^{\mathtt{G}}$.
- Classify ball maps. D'Angelo, M. Xiao, X. Huang, J. Faran, many others.
- Gap conditions.
  - $\mathtt{G} \geq \mathtt{g}$;
  - if $\mathtt{g} = \mathtt{G}$ then $\varphi$ is an autormorphism;
  - First gap thm: If $\mathtt{G} \leq 2\mathtt{g} - 1$ (and $\varphi$ is minimal), then $\mathtt{G} = \mathtt{g}$.
- Side questions; e.g., classify those equivalent to poly maps.

# Commutative Fock space ...
## ... and inner sequences

- *Drury-Arveson space** in dimension g is the RKHS with kernel

$$k_{\mathrm{g}}(z,w) = (1 - \langle z, w \rangle_{\mathbb{C}^{\mathrm{g}}})^{-1}.$$

- A (nice enough) function

$$\varphi = \begin{pmatrix} \varphi_1 & \dots & \varphi_m \end{pmatrix} : \mathbb{B}^{\mathrm{g}} \to \mathbb{B}^m$$

induces a mapping $M_\varphi : \oplus^m \mathscr{H}^2 \to \mathscr{H}^2$

$$M_\varphi \oplus h_j = \sum_{j=1}^{m} \varphi_j h_j.$$

- $\varphi$ is an *inner sequence* if $M_\varphi M_\varphi^* : \mathscr{H}^2 \to \mathscr{H}^2$ is a projection.
- inner sequences are ball maps.

# A Beurling Theorem

## ... and a minimal inner sequence

The *shifts* $S_j$ on $\mathscr{H}^2$ are the operators of multiplication $z_j$.

**Theorem.** If $\mathcal{M} \subseteq \mathscr{H}^2$ is (shift) invariant and $\dim \mathcal{M}^\perp < \infty$, then there is an inner sequence $\varphi : \mathbb{B}^g \to \mathbb{B}^m$ so that

(i) $\varphi : \mathbb{B}^g \to \mathbb{B}^m$ is a rational ball map;

(ii) $m$ is the *ball codimension* of $\mathcal{M}$ :

$$m = \dim \left[ \operatorname{span}\{1, z_j u : u \in \mathcal{M}^\perp\} \ominus \mathcal{M}^\perp \right]$$
$$= \dim \operatorname{span}\{1, z_j u : u \in \mathcal{M}^\perp\} - \dim \mathcal{M}^\perp$$

(iii) $P_\mathcal{M} = M_\varphi M_\varphi^*$ is the projection onto $\mathcal{M}$;

(iv) (minimality) if $\psi : \mathbb{B}^g \to \mathbb{B}^N$ and $P_\mathcal{M} = M_\psi M_\psi^*$, then $N \geq m$ and there is an isometry $V : \mathbb{C}^m \to \mathbb{C}^N$ such that $\psi = \varphi V^*$;

(v) characterized by a *realization like formula* (naturally nc in nature).

# A Beurling Theorem ...

## ... and the Ball codimension

**Theorem.** If $\mathcal{M}$ invariant and $\dim \mathcal{M}^\perp < \infty$, then, with

$$m = \dim \operatorname{span}\{1, z_j u : u \in \mathcal{M}^\perp\} - \dim \mathcal{M}^\perp$$

(the ball codimension) there is an inner sequence $\varphi : \mathbb{B}^{\mathrm{g}} \to \mathbb{B}^m$ such that

(ii) $\mathcal{M} = M_\varphi \oplus^m \mathscr{H}^2$; (iv′) $m$ is minimal;

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- $m \leq 1 + \mathrm{g} \dim \mathcal{M}^\perp - \dim \mathcal{M}^\perp = \dim \mathcal{M}^\perp (\mathrm{g} - 1) + 1$;
- (zero based invariant subspaces) If $\Lambda \subseteq \mathbb{B}^{\mathrm{g}}$ is finite and $\mathcal{M} = \{f \in \mathscr{H}^2 : f(\lambda) = 0\}$, then

$$m = \operatorname{ballcodim} \mathcal{M} = |\Lambda| (\mathrm{g} - 1) + 1 :$$

- If $\dim \mathcal{M}^\perp \geq 2$, then $m \geq 2\mathrm{g} - 1$ (compare first gap theorem);
- A degenerate case: $\mathcal{M}^\perp = [1, z_1, z_2]$ ($m = 3 < 3(2-1) + 1$.)

# A Beurling Theorem ...

## ... and the Ball codimension

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- A degenerate case: $\mathcal{M}^\perp = [1, z_1, z_2]$ $(m = 3 < 3(2-1) + 1.)$
- A guess: with $n = \operatorname{codim} \mathcal{M}$ and $m = \operatorname{ballcodim} \mathcal{M}$, if

$$\binom{\ell + g}{g} < n \le \binom{\ell + g + 1}{g},$$

then

$$\binom{\ell + g + 1}{g} \le m.$$

# Free (nc) Fock space ...

## ... an introduction

- Let $\zeta = (\zeta_1, \ldots, \zeta_{\mathrm{g}})$ denote *freely nc variables*;
- Let $<\zeta>$ denote the *words* (free monoid) in $\zeta$;
- $\mathbb{C}<\zeta>$ is the *free algebra*; e.g., $1 + \zeta_1\zeta_2 - 2\zeta_2\zeta_1$;
- *free Fock space* $\mathscr{F}^2_{\mathrm{g}} = \mathscr{F}^2$ is the Hilby space with basis $<\zeta>$.
- An $h \in \mathscr{F}^2$ (can be viewed as) is a formal power series

$$h = \sum_{w \in <\zeta>} h_w \, w$$

  with $\|h\|^2 = \sum_w |h_w|^2 < \infty$;
- Two sets of shifts:
  - $L = (L_1, \ldots, L_{\mathrm{g}})$ - the *left shifts* $L_j w = \zeta_j \, w$;
  - $R = (R_1, \ldots, R_{\mathrm{g}})$ - the *right shifts* $R_j w = w \, \zeta_j$.

# Shift invariant subspaces ...
## ... and multipliers of $\mathscr{F}^2$

- The maps on $\mathscr{F}^2$ by $L_j h = \zeta_j h = \sum_{w \in <\zeta>} h_w \zeta_j w$ are the *left shifts* - isometries
- A $f \in \mathscr{F}^2$ is a *right multiplier* of $\mathscr{F}^2$ if the densely defined map $R_f$ on $\mathbb{C}<\zeta>$ determined by $R_f v = v f$ extends to a bounded operator.
- Given $F = (F_0, F_1, \ldots, F_m)$ right multipliers, define $R_F : \oplus^m \mathscr{F}^2 \to \mathscr{F}^2$ by

$$R_F \oplus^m h_j = \sum_{j=1}^{m} R_{F_j} h_j.$$

- $\overline{\text{ran}} R_F = \overline{\text{ran}} R_F R_F^*$ is *left* (shift) invariant.

# Beurling for left shift invariant subspaces ...
## ... set free

**Theorem.** [DP, P, AP, many others] If $\mathcal{M} \subseteq \mathscr{F}^2$ is left invariant and $\mathcal{M}^\perp$ has dimension $n$, then, with $m = n(\mathrm{g}-1)+1$, there exists $F = (F_0, F_1, \dots, F_m)$ right multipliers such that

$$P_{\mathcal{M}} = R_F R_F^*.$$

Moreover, $\{F_0, F_1, \dots, F_m\} \subseteq \mathscr{F}^2$ is an orthonormal basis for the (wandering) subspace

$$\mathcal{M} \ominus (\oplus L_j \mathcal{M}) = \operatorname{span}\{1, L_j \mathcal{M}^\perp\} \ominus \mathcal{M}^\perp.$$

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- $m$ is minimal;
- $FF^*$ has trace one on the *boundary* (needs an interpretation);
- There is realization inspired formula characterization for such $F$.

# An nc Ball

## evaluations

- Given a tuple $X = (X_1, \ldots, X_{\mathrm{g}}) \in M_n(\mathbb{C})^{\mathrm{g}}$ and

$$w = \zeta_{j_1} \zeta_{j_2} \cdots \zeta_{j_k} \in <\zeta>,$$

  let

$$X^w = X_{j_1} X_{j_2} \cdots X_{j_k} = w(X).$$

- Let $\mathbb{B}^{\mathrm{nc}}_{\mathrm{g}}[n]$ denote the set of $X \in M_n(\mathbb{C})^{\mathrm{g}}$ such that

$$\sum_{n=0}^{\infty} \sum_{|w|=n} X^w X^{*w} = \sum_{w \in <\zeta>} X^w X^{*w}$$

  converges.
- An $h \in \mathscr{F}^2$ *evaluates* at an $X \in \mathbb{B}^{\mathrm{nc}}_{\mathrm{g}}$ as

$$E_X(h) = h(X) = \sum_{w \in <\zeta>} h_w X^w.$$

# An nc Ball
## and free points

- Given $Y \in \mathbb{B}^{\mathrm{nc}}_{\mathrm{g}}[n]$ and $y \in \mathbb{C}^n$, the pair $\mathbf{Y} = (Y, y)$ is a *free point of size n* if $y$ is cyclic for $Y$.
- If $\mathbf{Y}$ is a free point of size $n$, then

$$\mathscr{F}^2[\mathbf{Y}] = \{h \in \mathscr{F}^2 : h(Y)\, y = 0\}$$

is left invariant (invariant for $L$) and has co-dimension $n$ :

- the converse holds. Proof: Given $\mathcal{M}$, let $Y = L^*|_{\mathcal{M}^\perp}$ and choose $y = P_{\mathcal{M}^\perp} 1$. Show $\mathcal{M} = \mathscr{F}^2[\mathbf{Y}]$.
- A free point $\mathbf{Y}$ determines a free rational function,

$$G(\zeta) = y^* \big(I - \sum \zeta_j Y_j^*\big)^{-1} \Delta^{-1}(\mathbf{Y}).$$

where $\Delta^2(\mathbf{Y}) = \sum_{w \in <\zeta>} Y^w y y^* Y^{*w}$ and $\zeta_j \leftrightarrow Y_j^*$.

# Realization and Beurling

Given $\mathbf{Y} = (Y, y)$ of size $n$, let $\mathscr{F}^2[\mathbf{Y}] = \{f \in \mathscr{F}^2 : f(Y)\, y = 0\}$ and

$$G(\zeta) = y^* \left(I - \zeta_j Y_j^*\right)^{-1} \Delta^{-1}(\mathbf{Y}), \quad \Delta^2(\mathbf{Y}) = \sum_{w \in <\zeta>} Y^w y y^* Y^{*w}$$

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**Theorem.** For a canonical isometry $\mathscr{V} \in M_{ng+1, n(g-1)+1}$ and

$$F(\zeta) = \begin{pmatrix} 1 & \zeta_1 G(\zeta) & \dots & \zeta_g G(\zeta) \end{pmatrix} \mathscr{V} = \begin{pmatrix} F_0 & F_1 & \cdots & F_{n(g-1)} \end{pmatrix},$$

we have

$$P_{\mathscr{F}^2[\mathbf{Y}]} = R_F R_F^*.$$

- $\{F_j\}$ is an orthonormal basis for $\mathcal{M} \ominus \bigvee L_j \mathcal{M}$.
- $\dim \mathcal{M} \ominus \bigvee L_j \mathcal{M} = n(g-1) + 1$.

# Realization and Beurling

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we have

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- An analogous construction - with $\mathscr{V}$ not necessarily an isometry - in the commutative case gives a minimal inner function.
- Alternately, in the commutative case $Y$ is a commutative tuple and the commutative collapse of $F$ is the desired ball map.

# Boundary values, the column ball ...
## ... and evaluations

- The *column ball* $\mathscr{C} = (\mathscr{C}_{\mathrm{g}}[k])_k$ is the sequence of sets

$$\mathscr{C}_{\mathrm{g}}[k] = \{X = (X_1, \dots, X_{\mathrm{g}}) \in M_k(\mathbb{C})^{\mathrm{g}} : \sum X_j^* X_j \prec I\}.$$

- Given $X \in \mathscr{C}_{\mathrm{g}}$,

$$G(X) = \sum_{w \in <\zeta>} X^w \otimes y^* Y^{*w} \Delta^{-1}(\mathbf{Y}).$$

- Likewise, set

$$F(X) = \begin{pmatrix} I & X_1 G(X) & \cdots & X_{\mathrm{g}} G(X) \end{pmatrix} (I \otimes \mathscr{V}).$$

- $F$ is a free analytic map $\mathscr{C}_{\mathrm{g}} \to M(\mathbb{C})^{n(\mathrm{g}-1)+1}$.

# Boundary values, the column Ball ...
## ... and proper maps

- The (topological) boundary of $\mathscr{C}_{\mathtt{g}}$ consists of those $X$ such that $\ker(I - \sum X_j^* X_j) \neq \{0\}$;
- If $\Psi : \mathscr{C}_{\mathtt{g}} \to \mathscr{C}_{\mathtt{G}}$ is a proper free analytic (e.g. rational) map, then $\Psi$ is one-one (in fact essentially the restriction of an automorphism of $\mathscr{C}_{\mathtt{G}}$).
- In particular, our $F$ with $R_F R_F^* = P_{\mathscr{F}^2[\mathbf{Y}]}$ need not be proper.
- An $X \in M_k(\mathbb{C})^{\mathtt{g}}$ is in the *hard boundary* of $\mathscr{C}_{\mathtt{g}}$ if $\sum X_j^* X_j = I$.
- For $X$ of size $k$ in the hard boudary,

$$\operatorname{trace}(F(X)^* F(X)) = k.$$