Spaces of holomorphic, integrable k-differentials and Poincaré series map

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k-differentials

Some spaces of k-differentials

Poincaré series

Kernel of Poincaré series map

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- Equivalently, a k-differential on R is a collection of $\mathbb{C}$-valued functions $\{\phi_\alpha(z_\alpha)\}_{\alpha \in I}$ where z_α is a local complex coordinate on an open set U_α (for an open cover $\{U_\alpha\}_{\alpha \in I}$ of R), and for $\alpha, \beta \in I$ such that $U_\alpha \cap U_\beta \neq \emptyset$, we have:

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- The case $k = 2$ is of special importance: the 2-differentials (quadratic differentials) appear in Teichmüller Theory.

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- A k-differential is called holomorphic (meromorphic) if ϕ_α is holomorphic (meromorphic) for all $\alpha \in I$.

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  Let R be the Riemann sphere, with the conformal structure $U_1 = \mathbb{C}$ and $U_2 = \mathbb{C} \cup \{\infty\} \setminus \{0\}$. As parameters we introduce z and $w = \frac{1}{z}$. We can give an arbitrary function $\varphi_1(z)$ which is defined in whole plane and compute $\varphi_2(w)$ by transformation rule $\varphi_1(z)dz^k = \varphi_2(w)dw^k$, and get: $\varphi_2(w) = \varphi_1(\frac{1}{w})(\frac{-1}{w^2})^k$.

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Some spaces of holomorphic $k-$differentials

Proposition

(N.A., T. Foth) If Λ is a finite set, then:

- $A^{(2)}(V)$ is isomorphic to $A^{(2)}(\Sigma)$.
- $B(V)$ is isomorphic to $B(\Sigma)$.
- $A^{(1)}(V)$ is isomorphic to the space of integrable meromorphic k-differentials on Σ with at most simple poles in Λ.

Poincaré series

- The Poincaré series is $\theta(\phi) = \Sigma_{g \in \Gamma} \phi(gz)(\frac{d(gz)}{dz})^k$. It is convergent absolutely and uniformly on compact subset of Δ ($k > 1$ and $\phi \in A^{(1)}$).

Theorem

- *The Poincaré series map is defined by:*

$$\Theta : A^{(1)}(\Delta) \to A^{(1)}(\Sigma)$$

$$\phi(z)dz^k \mapsto \Sigma_{g \in \Gamma} \phi(gz)(\frac{d(gz)}{dz})^k dz^k.$$

- *It is linear, surjective and its norm is less than or equal to 1. (Quaziconformal Teichmüller Theory, F. Gardiner, N. Lakic)*

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- The set $\Theta_0(\{p(z)dz^k \mid p \in \mathbf{P}\})$ is dense in $A^{(1)}(U)$.

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- The proofs are application of the Poincaré series map.

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- Theorem
  (N.A., T. Foth) The set

  $$W = \{\beta(\chi_{g\mathcal{F}}\phi - \chi_{\gamma\mathcal{F}}\phi)(z)dz^k \mid g, \gamma \in \Gamma, \phi \in L^1_\Gamma(U, w^{k-2}d\mu)\},$$

  is dense in $\ker\Theta$.

- Theorem
  (N.A., T. Foth)
  Suppose Γ *is infinite. Let* P *be a subset of* $\mathcal{F}$ *that has a limit point in* $\mathcal{F}$. *Let* $\mathcal{P}$ *be the linear span of the set*

$$\{(K(z,p) - \overline{J(g,p)}^k K(z,gp))dz^k \mid p \in P, g \in \Gamma\}.$$

Then $\mathcal{P}$ *is in* $\ker\Theta \cap A^{(2)}(U)$ *and* $\mathcal{P}$ *is dense in* $A^{(2)}(U)$.

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- **Corollary**

  If Γ *is infinite then* $ker\Theta \cap A^{(2)}(U)$ *is dense in* $A^{(2)}(U)$.

Thank you !

Any question or comment?