Symbolic Manipulation of Harmonic Functions

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John Conway Day and SEAM

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Example: If $\zeta \in \mathbf{R}^n$ and $\|\zeta\| = 1$ then

$$\frac{1 - \|x\|^2}{\|x - \zeta\|^n}$$

is harmonic on $\mathbf{R}^n \setminus \zeta$.


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Dirichlet Problem: Suppose Ω is an open subset of $\mathbf{R}^n$. Given $f \in C(\partial\Omega)$, find $u \in C(\overline{\Omega})$ such that

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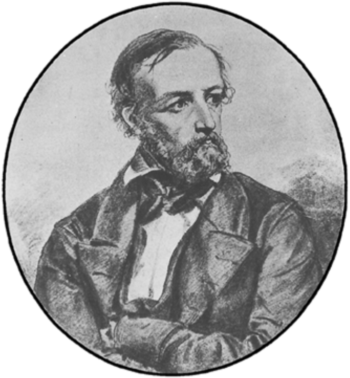

Johann Dirichlet (1805-1859)

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Theorem: Suppose $f \in C(\partial B)$. Define u on $\overline{B}$ by

$$u(x) = \begin{cases} \displaystyle\int_{\partial B} \frac{1 - \|x\|^2}{\|x - \zeta\|^n} f(\zeta)\, d\sigma(\zeta) & \text{if } x \in B \\ f(x) & \text{if } x \in \partial B. \end{cases}$$

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Surprising result: If f is a polynomial, then so is u.

Siméon Poisson (1781-1840)

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Thus $f + (1 - \|bx\|^2)g$ solves the Dirichlet problem on E with boundary function $f|_{\partial E}$.

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Let V_n denote volume measure on B_n.

Theorem Let f be a Borel measurable, integrable function on ∂B_n. If $1 \le k < n$, then

$$\int_{\partial B_n} f\, d\sigma_n =$$

$$\frac{k}{n} \frac{V(B_k)}{V(B_n)} \int_{B_{n-k}} (1 - |x|^2)^{\frac{k-2}{2}} \int_{\partial B_k} f(x, \sqrt{1 - |x|^2}\, \zeta)\, d\sigma_k(\zeta)\, dV_{n-k}(x).$$

Power series expansion of the Poisson kernel:

$$\int_{\partial B} \frac{1 - \|x\|^2}{\|x - \zeta\|^n} f(\zeta)\, d\sigma(\zeta)$$

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Theorem (Axler and Ramey, 1995): The Poisson integral of a polynomial f can be computed rapidly from

$$\mathcal{K}[D_f \|x\|^{2-n}].$$

Lord Kelvin (1824-1907)

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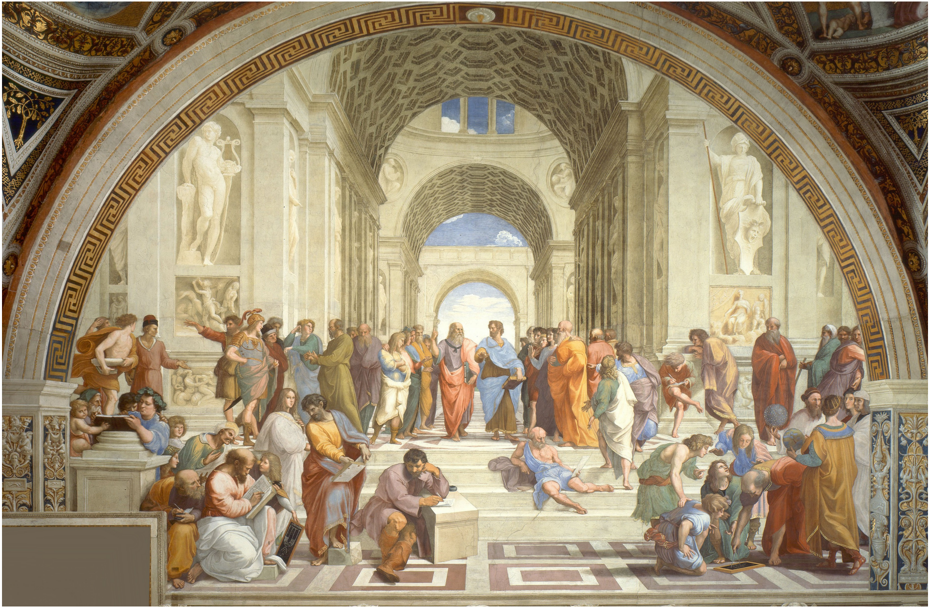
Theorem (Axler, Gorkin, and Voss, 2004): The solution to the Dirichlet problem for ellipsoids can be computed reasonably fast by repeated differentiation.

Link for software

The *Mathematica* package that implements these algorithms is available (without charge) at the following link:

`http://axler.net/HFT_Math.html`



Best wishes to John Conway!!!