The characteristic function as a unitary invariant for row contractions

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de Branges-Rovnyak model for c.n.c. T

Completely noncoisometric T

$T \in \mathcal{L}(\mathcal{X})$ c.n.c. : there exists no nonzero $\mathcal{M} \subset \mathcal{X}$ invariant for T^* with $T^*|_{\mathcal{M}}$ isometric

de Branges-Rovnyak kernel

Given $S \colon \mathbb{D} \underset{holo}{\to} \overline{\mathcal{B}}\mathcal{L}(\mathcal{U}, \mathcal{Y})$ ($S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$) define positive kernel K_S on $\mathbb{D}$: $K_S(z, \zeta) = [I - S(z)S(\zeta)^*]/(1 - z\overline{\zeta})$

de Branges-Rovnyak canonical model for c.n.c.

deBR model space = the RKHS $\mathcal{H}(K_S)$

deBR model operator: $T(S)$ on $\mathcal{H}(K_S)$ with $T(S)^* : f(z) \mapsto [f(z) - f(0)]/z$ (backward shift)

de Branges-Rovnyak canonical model for c.n.c.

The characteristic function of $T \in \overline{\mathcal{B}}\mathcal{L}(\mathcal{X})$

$\theta_T(z) = [-T + zD_{T^*}(I - zT)^{-1}D_T]|_{\mathcal{D}_T} : \mathcal{D}_T \to \mathcal{D}_{T^*}$
where $D_T = (I - T^*T)^{1/2}$, $\mathcal{D}_T = \overline{\text{Ran}}\, D_T$

The characteristic function as a complete unitary invariant:

Theorem: Given T, T' c.n.c., then $T \underset{u}{\cong} T' \Leftrightarrow \theta_T \underset{coincide}{\cong} \theta_{T'}$ and then T and $T' \underset{u}{\cong} T(\theta_T)$

Coincidence of Schur-class functions:

$S \underset{coincide}{\cong} S'$: there exist unitary $\alpha \colon \mathcal{Y} \to \mathcal{Y}'$ and $\beta \colon \mathcal{U}' \to \mathcal{U}$
so that $S'(z) = \alpha S(z) \beta$ for all $z \in \mathbb{D}$

Multivariable generalizations

Row contractions

$\mathbf{T} = (T_1, \dots, T_d) =$ **row contraction**: $\|T\| \leq 1$ where
$T = \begin{bmatrix} T_1 & \cdots & T_d \end{bmatrix} : \mathcal{X}^d \to \mathcal{X}$

Completely noncoisometric row contraction

$\mathbf{T}$ c.n.c.: there is no nonzero $\mathcal{M} \subset \mathcal{X}$ invariant for each T_j^* such that $\begin{bmatrix} T_1 & \cdots & T_d \end{bmatrix}^* |_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}^d$ is isometric

Characteristic function

Characteristic function: general noncommutative case

$\theta_{T,nc}(\mathbf{z}) = [-T + D_{T^*}(I - Z_{\mathcal{X}}(\mathbf{z})T^*)^{-1}Z_{\mathcal{X}}(\mathbf{z})D_T]|_{\mathcal{D}_T}$
$=$ formal power series in $\mathcal{L}(\mathcal{D}_T, \mathcal{D}_{T^*})\langle\langle \mathbf{z} \rangle\rangle$
($\mathbf{z} = (\mathbf{z}_1, \dots, \mathbf{z}_d)$ = freely noncommuting indeterminates)
(or: θ_T = function of noncommuting operators)

M_{θ_T} = contractive on Fock space (noncommutative Schur class)

Notation:

$Z_{\mathcal{X}}(\mathbf{z}) = \begin{bmatrix} \mathbf{z}_1 I_{\mathcal{X}} & \cdots & \mathbf{z}_d I_{\mathcal{X}} \end{bmatrix}$
$D_T = (I_{\mathcal{X}^d} - T^*T)^{1/2}$,
$\mathcal{D}_T \subset \overline{\text{Ran}}\, D_T \subset \mathcal{X}^d$,
$D_{T^*} = (I_{\mathcal{X}} - TT^*)^{1/2} = (I_{\mathcal{X}} - T_1T_1^* - \cdots - T_dT_d^*)^{1/2}$,
$\mathcal{D}_{T^*} = \overline{\text{Ran}}\, D_{T^*} \subset \mathcal{X}$

Expanding out gives: $\theta_T(\mathbf{z}) = \sum_{v \in \mathcal{F}_d} [\theta_{\mathbf{T},nc}]_v \mathbf{z}^v$
where $[\theta_{\mathbf{T},nc}]_\emptyset = -T|_{\mathcal{D}_T}$, $[\theta_{\mathbf{T},nc}]_{v'j} = D_{T^*}\mathbf{T}^{*v'}\mathcal{I}_j^* D_T$
($\mathbf{T}^{*v} = T_{i_N}^* \cdots T_{i_1}^*$ if $v = i_N \cdots i_1$)

The characteristic function as unitary invariant for c.n.c.

Theorem (Popescu): Given $\mathbf{T}$, $\mathbf{T}'$ c.n.c., then $\mathbf{T} \underset{u}{\cong} \mathbf{T}' \Leftrightarrow \theta_{T,nc} \underset{coincide}{\cong} \theta_{T',nc}$ and then $\mathbf{T}$, $\mathbf{T}' \underset{u}{\cong} \mathbf{T}(\theta_{T,nc})$

Multivariable generalizations: The commutative case

Commutative row contractions

$\mathbf{T} = (T_1, \ldots, T_d) =$ row contraction with T_j's commuting

$\mathbf{T}$ c.n.c.: same as above: no nonzero $\mathbf{T}^*$-invariant subspace on which T^* is isometric

The commutative characteristic function

$\theta_T(z) = [-T + D_{T^*}(I - Z_{\mathcal{X}}(z)T^*)^{-1} Z_{\mathcal{X}}(z) D_T]|_{\mathcal{D}_T} : \mathcal{D}_T \to \mathcal{D}_{T^*}$
where $z = (z_1, \ldots, z_d) \in \mathbb{B}^d$

$M_{\theta_T} =$ contractive multiplier on the Drury-Arveson space

The characteristic function as complete unitary invariant: commutative case

Theorem (Bhattacharyya-Eschmeier-Sarkar) Given $\mathbf{T}$, $\mathbf{T}'$ commutative c.n.c., then $\mathbf{T} \underset{u}{\cong} \mathbf{T}' \Leftrightarrow \theta_T \underset{coincide}{\cong} \theta_{T'}$ and then $\mathbf{T}$ and $\mathbf{T}' \underset{u}{\cong} \mathbf{T}(\theta_T)$

The classical case: completely nonunitary contractions

$T \in \mathcal{L}(\mathcal{X})$ c.n.u.: no T-reducing subspace $\mathcal{M}$ so that $T|_{\mathcal{M}}$ unitary

The Sz.-Nagy-Foias model

Given $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$, set $\mathcal{K}(S) = \begin{bmatrix} H^2_{\mathcal{Y}} \\ \Delta_S \cdot L^2_{\mathcal{U}} \end{bmatrix} \ominus \begin{bmatrix} S \\ \Delta_S \end{bmatrix} H^2_{\mathcal{U}}$

(where $\Delta_S(\zeta) = (I - S(\zeta)^* S(\zeta))^{1/2}$ for $\zeta \in \mathbb{T}$) and set

$T_{NF}(S) = P_{\mathcal{K}(S)} \begin{bmatrix} M_z & 0 \\ 0 & M_\zeta \end{bmatrix} \Big|_{\mathcal{K}(S)}$

The characteristic function as a complete unitary invariant

Theorem (Sz.Nagy-Foias): Given c.n.u. $T, T' \in \mathcal{L}(\mathcal{X})$, then $T \underset{u}{\cong} T'$ if and only if $\theta_T \underset{coincide}{\cong} \theta_{T'}$ and then T and $T' \underset{u}{\cong} T_{NF}(\theta_T)$

The de Branges-Rovnyak two-component kernel

Given $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$, set $\widehat{K}_S(z,\zeta) = \begin{bmatrix} \frac{I - S(z)S(\zeta)^*}{1 - z\overline{\zeta}} & \frac{S(z) - S(\overline{\zeta})}{z - \overline{\zeta}} \\ \frac{S(\overline{z})^* - S(\zeta)^*}{z - \overline{\zeta}} & \frac{I - S(\overline{z})^* S(\overline{\zeta})}{1 - z\overline{\zeta}} \end{bmatrix} =$ positive kernel on $\mathbb{D}$

The de Branges-Rovnyak model operator

Define $\widehat{T}_{dBR}(S)$ on $\mathcal{H}(\widehat{K}_S)$ by

$$\widehat{T}_{dBR}(S)^* \colon \begin{bmatrix} f(z) \\ g(z) \end{bmatrix} \mapsto \begin{bmatrix} [f(z) - f(0)]/z \\ zg(z) - S(z)f(0) \end{bmatrix}$$

The characteristic function as complete unitary invariant

Theorem (de Branges-Rovnyak): Given c.n.u. $T, T' \in \mathcal{L}(\mathcal{X})$, then $T \underset{u}{\cong} T'$ if and only if $\theta_T \underset{coincide}{\cong} \theta_{T'}$ and then T and $T' \underset{u}{\cong} \widehat{T}_{dBR}(\theta_T)$

Beyond c.n.c.: multivariable setting

Noncommutative c.n.u.

$\mathbf{T} = (T_1, \dots, T_d)$ **c.n.u.**: there is no nonzero $\mathcal{M} \subset \mathcal{X}$ reducing for each T_j such that $T|_{\mathcal{M}^d} \colon \mathcal{M}^d \to \mathcal{M}$ is unitary.

Amusing fact:

If $d > 1$ and $\mathcal{X} \neq \{0\}$, then any commutative row contraction is c.n.u.

Unitary invariants for c.n.u. row contraction:

Theorem (B.-Vinnikov): Given $\mathbf{T}$ c.n.u., besides $\theta_{T,nc}$ there is an additional invariant L_T (a noncommutative analogue of $\Delta_{\theta_T}(\zeta) = I - \theta_T(\zeta)^* \theta_T(\zeta)$ but not uniquely determined by $\theta_{T,nc}$) so that, given c.n.u. T, T', then $T \underset{u}{\cong} T' \Leftrightarrow (\theta_{T,nc}, L_T) \underset{coincide}{\cong} (\theta_{T',nc}, L_{T'})$ and then T and $T' \underset{u}{\cong} T(\theta_T, L_T)$

= noncommutative Sz.-Nagy-Foias model

Multivariable commutative two-component dBR model

Drury-Arveson asymmetry

For S = Drury-Arveson-space contractive multiplier,
$K_S(z,\zeta) = \frac{I - S(z)S(\zeta)^*}{1-\langle z,\zeta\rangle}$ is a positive kernel on $\mathbb{B}^d$ but
$\widetilde{K}_S(z,\zeta) = \frac{I - S(z)^*S(\zeta)}{1-\langle \zeta, z\rangle}$ may not be a positive kernel

Required analogue: Agler decomposition:

There exists a positive kernel $\Phi = [\Phi_{ij}]_{i,j=1,\dots,d}$ so that
$I - S(z)^*S(\zeta) = \sum_{j=1}^d \Phi_{jj}(z,\zeta) - \sum_{i,\ell=1}^d \overline{z}_i \zeta_\ell \Phi_{i\ell}(z,\zeta)$

Two-component Agler decomposition

Analogue of two-component deBR kernel

The kernel K_S extends to a positive kernel

$$\mathbb{K}(z,\zeta) = \begin{bmatrix} K_S(z,\zeta) & \Psi_1(z,\zeta) & \cdots & \Psi_d(z,\zeta) \\ \Psi_1(\zeta,z)^* & \Phi_{11}(z,\zeta) & \cdots & \Phi_{1d}(z,\zeta) \\ \vdots & \vdots & & \vdots \\ \Psi_d(\zeta,z)^* & \Phi_{d1}(z,\zeta) & \cdots & \Phi_{dd}(z,\zeta) \end{bmatrix} \text{ on } \mathbb{B}^d \text{ such that}$$

$$\begin{bmatrix} I \\ S(z)^* \end{bmatrix} \begin{bmatrix} I & S(\zeta) \end{bmatrix} - \begin{bmatrix} S(z) \\ I \end{bmatrix} \begin{bmatrix} S(\zeta)^* & I \end{bmatrix} =$$

$$M(z)^* \mathbb{K}(z,\zeta) M(\zeta) - \sum_{j=1}^d N_j(z)^* \mathbb{K}(z,\zeta) N_j(\zeta)$$

where $M(z) = \begin{bmatrix} I_{\mathcal{Y}} & 0 \\ 0 & Z_{\mathcal{U}}(\overline{z})^* \end{bmatrix}$, $N_j(z) = \begin{bmatrix} z_j I_{\mathcal{Y}} & 0 \\ 0 & \mathbf{e}_j \otimes I_{\mathcal{U}} \end{bmatrix}$

Model Agler decomposition:

$S = \theta_T$ and $\mathbb{K}_T(z,\zeta) = \mathbb{G}_T(z)\mathbb{G}_T(\zeta)^*$ with

$$\mathbb{G}_T(z) = \begin{bmatrix} D_{T^*}(I - Z_{\mathcal{X}}(z)T^*)^{-1} \\ D_T(I - Z_{\mathcal{X}}(z)^*T)^{-1}\mathcal{I}_1 \\ \vdots \\ D_T(I - Z_{\mathcal{X}}(z)^*T)^{-1}\mathcal{I}_d \end{bmatrix}$$

Two-component canonical functional model

$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H}(\mathbb{K}) \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}(\mathbb{K})^d \\ \mathcal{Y} \end{bmatrix}$ = **t.c.f.m.** colligation associated with Agler decomposition $\mathbb{K}$ for $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$:

- $A = \text{col}_{1 \le k \le d} A_k$ solves structured Gleason problem $(\mathbf{s}f)(z) - (\mathbf{s}f)(0) = \sum_{k=1}^d z_k (A_k f)_+(z)$ for all $f \in \mathcal{H}(\mathbb{K})$
- A^* solves the dual structured Gleason problem $(\widetilde{\mathbf{s}}g)(z) - (\widetilde{\mathbf{s}}g)(0) = \sum_{k=1}^d \overline{z}_k (A^* g)_{-,k}(z)$ for all $g \in \mathcal{H}(\mathbb{K})^d$
- $C \colon f \mapsto (\mathbf{s}f)(0)$, $B^* \colon g \mapsto (\widetilde{\mathbf{s}}g)(0)$, $D \colon u \mapsto S(0)u$

A, A^* = multivariable analogues of backward shift operators

Here

- $f \in \mathcal{H}(\mathbb{K})$ has the form $f = \begin{bmatrix} f_+ \\ f_{-,1} \\ \vdots \\ f_{-,d} \end{bmatrix}$ with $\mathbf{s}f = f_+$,
- $g \in \mathcal{H}(\mathbb{K})^d$ has the form $g = \begin{bmatrix} g_1 \\ \vdots \\ g_d \end{bmatrix}$ with each $g_i = \begin{bmatrix} g_{i,+} \\ g_{i,-,1} \\ \vdots \\ g_{i,-,d} \end{bmatrix}$

  and $\widetilde{\mathbf{s}}g = \sum_{i=1}^d g_{i,-,i}$

Classes of row contractions

- **commutative closely connected (com c.c.)**:
  $\mathcal{M}_T^{(1)} := \{x \in \mathcal{X} \colon D_{T^*}(I - Z_{\mathcal{X}}(z)T^*)^{-1}x \equiv 0$
  and $D_T(I - Z_{\mathcal{X}}(z)^* T)^{-1}\mathcal{I}_i x \equiv 0$ for $i = 1, \dots, d\} = \{0\}$
- **strongly closely connected (strongly c.c.)**:
  $\mathcal{M}_T^{(2)} := \{x \in \mathcal{X} \colon D_{T^*}(I - Z_{\mathcal{X}}(z)T^*)^{-1}x \equiv 0$
  and $D_T(I - Z_{\mathcal{X}}(z)^* T)^{-1} Z_{\mathcal{X}}(z)^* x \equiv 0\} = \{0\}$

Summary:

operator d-tuple class	complete unitary invariant
$\mathbf{T} =$ commutative **c.n.c.**	θ_T
$\mathbf{T} =$ commutative **strongly c.c.**	$(\theta_T, \mathbb{K}_T)$
$\mathbf{T} =$ commutative **c.c.**	$(\theta_T, \mathbb{K}_T, X_T)$
$\mathbf{T} =$ **c.n.u.**	$(\theta_{T,nc}, \mathbb{K}_{T,nc})$

$\mathbb{K}_{T,nc}$ = noncommutative Agler decomposition:
de Branges-Rovnyak noncommutative model = reinterpretation of

Example: Spherical isometries

$T = [\lambda_1 \ \lambda_2]$, $(\lambda_1, \lambda_2) \in \mathbb{S}^2 = \partial \mathbb{B}^2$ = strongly **cc** case

$D_{T^*} = 0 \colon \mathbb{C}^2 \to \{0\} \Rightarrow \theta_T(z) \equiv 0 \colon \mathcal{D}_T \cong \mathbb{C} \to \{0\}$ for all such λ

$\mathbb{K}_\lambda(z, \zeta) = \mathbb{G}_\lambda(z) \mathbb{G}_\lambda(\zeta)^*$ where $G_\lambda(z) = \frac{1}{1 - \overline{z}_1 \lambda_1 - \overline{z}_2 \lambda_2} \begin{bmatrix} \overline{z}_2 - \overline{\lambda}_2 \\ -\overline{z}_1 + \overline{\lambda}_1 \end{bmatrix}$
= the extra invariant!

Agler decomposition (after clearing out denominator):

$$
\begin{aligned}
&(1 - \overline{z}_1 \lambda_1 - \overline{z}_2 \lambda_2)\left(1 - \zeta_1 \overline{\lambda}_1 - \zeta_2 \overline{\lambda}_2\right) = \\
&\quad (\overline{z}_2 - \overline{\lambda}_2)(\zeta_2 - \lambda_2) + (\overline{z}_1 - \overline{\lambda}_1)(\zeta_1 - \lambda_1) \\
&\quad - \overline{z}_1 \zeta_1 (\overline{z}_2 - \overline{\lambda}_2)(\zeta_2 - \lambda_2) + \overline{z}_1 \zeta_2 (\overline{z}_2 - \overline{\lambda}_2)(\zeta_1 - \lambda_1) \\
&\quad + \overline{z}_2 \zeta_1 (\overline{z}_1 - \overline{\lambda}_1)(\zeta_2 - \lambda_2) - \overline{z}_2 \zeta_2 (\overline{z}_1 - \overline{\lambda}_1)(\zeta_1 - \lambda_1)
\end{aligned}
$$

Amusing exercise: Check directly