Finite Math

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IAMS 2011


hello.


SONY
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Collaborators

Angiuli
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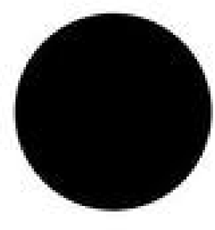
Artist's conception of a II_1 factor



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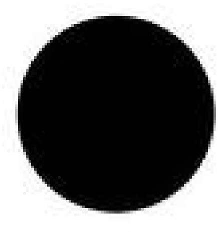

"Some detail is lost in the fine print." — *V.F.R. Jones*

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- $x = x^*, y = y^* \in A$
  $\{x, y\}'' \Longleftrightarrow \{\tau(p(x, y)) : p \in \mathbb{C}\langle X, Y\rangle\}$

Microstates

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  Also known as the CEP

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- Equivalent formulation:

  $\exists ?\ x' = x'^*, y' = y'^* \in M_\omega$ with $(x', y') =_D (x, y)$

- Q(n): $\forall \alpha = \alpha^*, \beta = \beta^* \in M_n(\mathbb{C})\ \exists x', y' \in M_\omega$ with

$$x =_D x', y =_d y' \text{ and } \alpha \otimes x + \beta \otimes y =_D \alpha \otimes x' + \beta \otimes y'$$

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may assume $\alpha = \beta = 1$

Flags

- $(e_t)_{t\in[0,1]} \subset A$ increasing projections, $\tau(e_t) = t$
- $x = x^* \in A$

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$\exists X_n, Y_n \in M_n(\mathbb{C})$ s.a. so eigenvalues of X_n $(Y_n, X_n + Y_n)$ are α_j (β_j, γ_j).
(Hence Q(1).)

- (Klyachko) Must check inequalities

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$$1 - \frac{2}{n} + 1 - \frac{3}{n} + \frac{6}{n} = \frac{1}{n},$$

just right. (Try the second inequality.)

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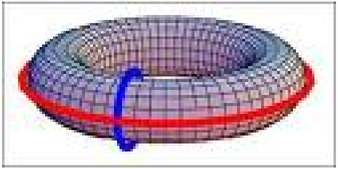
  Intersection numbers

  classical and "quantum"



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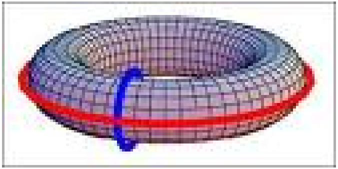

- classical = #of intersection points
- Quantum: rational maps $h : S^2 \to G(n, r)$, given degree, so $h(P_j) \in \mathfrak{S}_j$

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- (Belkale, Woodward-Agnihotri) Yes if we can prove even more inequalities. Not all homogeneous.

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- $(\alpha \otimes a)(\beta \otimes b)$; Perhaps CEP $\Leftrightarrow$ $Q^\times(1)$.

Thanks!