Differentiating Matrix Functions

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The Functional Calculus

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Then, $F : S_n \to S_n$.

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Then, we can differentiate using the product rule.

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For $t \neq 0$, the eigenvalues of $S(t)$ are $\pm\, e^{-\frac{1}{t^2}}$ and their associated eigenvectors are:

$$\pm \begin{pmatrix} \cos(\frac{1}{t}) \\ \sin(\frac{1}{t}) \end{pmatrix} \text{ and } \pm \begin{pmatrix} \sin(\frac{1}{t}) \\ -\cos(\frac{1}{t}) \end{pmatrix}.$$

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Thus, even an infinitely differentiable curve can have discontinuous eigenvectors.

One-Variable Results

Define the spaces:

$$
\begin{aligned}
C^m(\mathbb{R},\mathbb{R}) &= \{f : \mathbb{R} \to \mathbb{R} : f \text{ is } m\text{-times continuously differentiable}\} \\
C^m(S_n, S_n) &= \{F : S_n \to S_n : F \text{ is } m\text{-times continuously Frechét differentiable}\}
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S_n(K) = \{n \times n \text{ self -adjoint matrices with spectrum in } K\}.
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Theorem 1 (Brown and Vasudeva, 2000)

If f is in $C^m(\mathbb{R},\mathbb{R})$, then the induced matrix function F is in $C^m(S_n,S_n)$.

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- $C^m(\mathbb{R},\mathbb{R})$ and $C^m(S_n,S_n)$ are Frechét spaces induced by the semi-norms

$$\|f^{(l)}\|_K = \sup_{x\in K} |f^{(l)}(x)| \quad \text{and} \quad \|d^l F\|_K = \sup_{A\in S_n(K)} \|d^l F(A)\|,$$

for $0 \leq l \leq m$ and $K \subset \mathbb{R}$ compact.

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$$\|d^l P\|_K \leq n!l!\|p^{(l)}\|_K$$

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$$\{p_j\} \to f \quad \text{in } C^m(\mathbb{R},\mathbb{R}).$$

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- Then, $\{P_j\}$ is Cauchy and hence, converges to some $G \in C^m(S_n,S_n)$. It can be shown that $G = F$.

Two-Variable Considerations

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Lemma 1

Let $S(t) = (S_1(t), S_2(t))$ be a curve in CS_n. If $S(t)$ is Lipschitz, the joint eigenvalues of $S(t)$ can be represented by Lipschitz functions

$$(x_i(t), y_i(t)) \qquad \text{for } i = 1, \ldots, n.$$

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- For ease of notation, define:

$$\begin{aligned} U &:= U(t) & \\ (x_i, y_i) &:= (x_i(t), y_i(t)) & \text{for } i = 1, \dots, n. \\ S_r' &:= S_r'(t) & \text{for } r = 1, 2. \end{aligned}$$

Divided Differences

Let $f \in C^1(\mathbb{R}, \mathbb{R})$. The **first divided difference** of f is defined by

$$f^{[1]}(a,b) = \frac{f(a) - f(b)}{a - b} \quad \text{for } a \neq b$$

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Two-variable Results: Existence

Theorem 2 (B., 2010)

If $f \in C^1(\mathbb{R}^2, \mathbb{R})$ and $S(t)$ is a C^1 curve in CS_n, then $\frac{d}{dt}F(S(t))$ exists and

$$\begin{aligned} \frac{d}{dt}F(S(t)) &= U\Big(\big[f^{[1,0]}(x_i, x_j; y_j)\big]_{i,j=1}^{n} \odot \big(U^* \; S_1' \; U\big) \\ &\quad + \big[f^{[0,1]}(x_i; y_i, y_j)\big]_{i,j=1}^{n} \odot \big(U^* \; S_2' \; U\big)\Big) U^*, \end{aligned} \tag{1}$$

where $f^{[1,0]}$ and $f^{[0,1]}$ are divided differences taken in the first and second variables respectively and $\odot$ denotes the Schur product.

Lemma 2

If f is a real-valued analytic function on $\mathbb{R}^2$ and $S(t)$ is a C^1 curve in CS_n, then $\frac{d}{dt}F(S(t))$ exists, is of form (1), and is continuous.

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If f is a real-valued analytic function on $\mathbb{R}^2$ and $S(t)$ is a C^1 curve in CS_n, then $\frac{d}{dt}F(S(t))$ exists, is of form (1), and is continuous.

Proof Sketch

Fix t^*. For all t sufficiently close to t^*:

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Fix t^*. For all t sufficiently close to t^*:

$$F(S(t)) = \frac{1}{(2\pi i)^2} \int_{C_2} \int_{C_1} f(\zeta_1, \zeta_2) \left(\zeta_1 I - S_1(t)\right)^{-1} \left(\zeta_2 I - S_2(t)\right)^{-1} d\zeta_1 d\zeta_2,$$

where C_1 and C_2 are curves containing the eigenvalues of $S_1(t^*)$ and $S_2(t^*)$.

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where C_1 and C_2 are curves containing the eigenvalues of $S_1(t^*)$ and $S_2(t^*)$.

Since the integrand is bounded near t^*, we can differentiate under the integral.

The result is a continuous function of t.

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Fix t^*. Choose a polynomial p so that p and f agree to first order on the joint eigenvalues of $S(t^*)$.

Recall that the eigenvalue functions (x_i, y_i), for $i = 1, \ldots, n$, are Lipschitz.

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Recall that the eigenvalue functions (x_i, y_i), for $i = 1, \ldots, n$, are Lipschitz.Then

$$\begin{aligned}\|F(S(t)) - P(S(t))\| &= \max_{1\le i\le n} |f(x_i, y_i) - p(x_i, y_i)| \\ &= \max_{1\le i\le n} \Big|(f-p)(x_i, y_i) - (f-p)\big(x_i(t^*), y_i(t^*)\big)\Big| \\ &= o\big(|t - t^*|\big),\end{aligned}$$

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Thus, $F(S(t))$ is differentiable at t^* and

$$\frac{d}{dt} F(S(t))\Big|_{t=t^*} = \frac{d}{dt} P(S(t))\Big|_{t=t^*}. \qquad \square$$

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If $f \in C^1(\mathbb{R}^2, \mathbb{R})$ and $S(t)$ is a C^1 curve in CS_n, then $\frac{d}{dt}F(S(t))$ is continuous.

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Proof

Fix t_0. For every $g \in C^1(\mathbb{R}^2, \mathbb{R})$ and t^* sufficiently close to t_0,

$$\left\| \frac{d}{dt} G(S(t))|_{t=t^*} \right\| \leq C \sup_{(x,y)\in K} \{|g_x(x,y)|, |g_y(x,y)|\},$$

for a fixed constant C and compact set K.

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for a fixed constant C and compact set K.

Let $f \in C^1(\mathbb{R}^2, \mathbb{R})$. There exists a sequence $\{\phi_m\}$ of analytic functions such that:

$$\{\phi_m\} \to f \text{ uniformly on } K$$

and

$$\sup_{(x,y)\in K} \left\{\left|(\phi_m - f)_x(x,y)\right|, \left|(\phi_m - f)_y(x,y)\right|\right\} \leq \tfrac{1}{m}.$$

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Then for all t^* sufficiently close to t_0,

$$\left\| \frac{d}{dt}\Phi_m(S(t))\Big|_{t=t^*} - \frac{d}{dt}F(S(t))\Big|_{t=t^*} \right\| \leq \frac{C}{m},$$

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By Lemma 1, each $\frac{d}{dt}\Phi_m(S(t))$ is continuous at each t^*.

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By Lemma 1, each $\frac{d}{dt}\Phi_m(S(t))$ is continuous at each t^*. Thus, $\frac{d}{dt}F(S(t))$ is continuous in a neighborhood of t_0.

□

Generalizations

- Let $f \in C^1(\mathbb{R}^d, \mathbb{R})$ and define

  $CS_n^d :=$ { d-tuples of pairwise commuting $n \times n$ self-adjoint matrices}.

  If $S(t)$ is a C^1 curve in CS_n^d, then

$$\frac{d}{dt} F(S(t)) \text{ exists and is continuous.}$$

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- Let $f \in C^1(\mathbb{R}^d, \mathbb{R})$ and define

  $CS_n^d :=$ { d-tuples of pairwise commuting $n \times n$ self-adjoint matrices}.

  If $S(t)$ is a C^1 curve in CS_n^d, then

  $$\frac{d}{dt} F(S(t)) \text{ exists and is continuous.}$$

- Let $f \in C^m(\mathbb{R}^2, \mathbb{R})$ and $S(t)$ be a C^m curve in CS_n.

  Then

  $$\frac{d^m}{dt^m} F(S(t)) \text{ exists and is continuous.}$$

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