Schroeder's Equation in Several Variables

Robert A. Bridges
bridges@purdue.edu

Purdue University
West Lafayette, IN

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Schroeder's Equation for $n = 1$

Given analytic $\phi : \mathbb{D} \to \mathbb{D}$ such that

- $\phi(0) = 0$.
- $\phi \neq 0$.

Does there exist an analytic $f : \mathbb{D} \to \mathbb{C}$ such that

- $f \circ \phi = \phi'(0) f$ or $C_\phi f = \phi'(0) f$
- $f \neq 0$?

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Theorem (Koenig 1884)

YES! if $0 < |\phi'(0)| \leq 1$

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Proof: $\dfrac{\phi^{(n)}(z)}{\phi'(0)^n} \to f(z)$

2003 Cowen & MacCluer generalize to $\mathbb{B}^n$

Given $\phi : \mathbb{B}^n \to \mathbb{B}^n$
such that

- ϕ analytic
- $\phi(0) = 0$

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Given $\phi : \mathbb{B}^n \to \mathbb{B}^n$
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- $\phi(0) = 0$
- $\phi'(0)$ full rank

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Given $\phi : \mathbb{B}^n \to \mathbb{B}^n$ such that

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- $\phi(0) = 0$
- $\phi'(0)$ full rank
- $|\phi(z)| < |z|$ for $0 < |z| < 1$

Does there exist $F : \mathbb{B}^n \to \mathbb{C}^n$ such that

- F is analytic
- $F \circ \phi = \phi'(0)F$

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- F is analytic
- $F \circ \phi = \phi'(0)F$
- F has full rank near 0.

C & M give necessary and sufficient conditions under the additional hypothesis

- $\phi'(0)$ diagonalizable.

Notation

Represent functions as (infinite) column vectors with entries from their Taylor Series; for example

$$f(z_1, z_2) = a_{0,0} + a_{1,0}z_1 + a_{0,1}z_2 + a_{2,0}z_1^2 + \ldots = \begin{bmatrix} a_{0,0} \\ a_{1,0} \\ a_{0,1} \\ \vdots \end{bmatrix}.$$

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We want to represent C_ϕ as a large (infinite) matrix. Let $\phi = (\phi_1, \phi_2)$ with

$$\phi_1 = \begin{bmatrix} 0 \\ b_{1,0} \\ b_{0,1} \\ \vdots \end{bmatrix}, \text{ and } \phi_2 = \begin{bmatrix} 0 \\ c_{1,0} \\ c_{0,1} \\ \vdots \end{bmatrix}.$$

Notation Continued

Since C_ϕ maps

$$\begin{aligned} z_1 &\to \phi_1, \\ z_2 &\to \phi_2, \\ z_1^2 &\to \phi_1^2, \\ &\vdots \end{aligned}$$

$$C_\phi = \begin{array}{c} \\ 1 \\ z_1 \\ z_2 \\ z_1^2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} 1 & \phi_1 & \phi_2 & \phi_1^2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & b_{1,0} & c_{1,0} & 0 & \dots \\ 0 & b_{0,1} & c_{0,1} & 0 & \dots \\ 0 & b_{2,0} & c_{2,0} & b_{1,0}^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}.$$

Notation Continued

$$C_\phi = \begin{matrix} & 1 & \phi_1 & \phi_2 & \phi_1^2 & \\ 1 & 1 & 0 & 0 & 0 & \dots \\ z_1 & 0 & b_{1,0} & c_{1,0} & 0 & \dots \\ z_2 & 0 & b_{0,1} & c_{0,1} & 0 & \dots \\ z_1^2 & 0 & b_{2,0} & c_{2,0} & b_{1,0}^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix} .$$

Notice that

- C_ϕ is block lower triangular
- After omitting the first row and column of C_ϕ, $\phi'(0)^t$ is the upper left $n \times n$.

Jordan form glasses

$$\phi'(0) = \begin{bmatrix} \lambda & 1 & & & \\ 0 & \lambda & & & \\ & & \alpha & 1 & 0 \\ & & 0 & \alpha & 1 \\ & & 0 & 0 & \alpha \end{bmatrix},$$

Jordan form glasses

$$\phi'(0) = \begin{bmatrix} \lambda & 1 & & & \\ 0 & \lambda & & & \\ & & \alpha & 1 & 0 \\ & & 0 & \alpha & 1 \\ & & 0 & 0 & \alpha \end{bmatrix},$$ if and only if $\phi'(0)^t$ has two chains, namely

$(\phi'(0) - \lambda I_n) : e_2 \mapsto e_1 \mapsto 0$, and

$(\phi'(0) - \alpha I_n) : e_5 \mapsto e_4 \mapsto e_3 \mapsto 0$.

Jordan form glasses

$$\phi'(0) = \begin{bmatrix} \lambda & 1 & & & \\ 0 & \lambda & & & \\ & & \alpha & 1 & 0 \\ & & 0 & \alpha & 1 \\ & & 0 & 0 & \alpha \end{bmatrix},$$

Schroeder's equation is $\begin{bmatrix} C_\phi f_1 \\ \vdots \\ C_\phi f_5 \end{bmatrix} = \phi'(0) \begin{bmatrix} f_1 \\ \vdots \\ f_5 \end{bmatrix} = \begin{bmatrix} \lambda f_1 + f_2 \\ \lambda f_2 \\ \lambda f_3 + f_4 \\ \lambda f_4 + f_5 \\ \lambda f_5 \end{bmatrix}$

$(C_\phi - \lambda I) : f_1 \mapsto f_2 \mapsto 0$ and
$(C_\phi - \alpha I) : f_3 \mapsto f_4 \mapsto f_5 \mapsto 0.$

Theorem

There exists a solution to Schroeder's equation $\Leftrightarrow$ given any chain of $\phi'(0)$, there is a chain of C_ϕ of greater length with the same eigenvalue.

Outline

1. Reduce to $\phi'(0) = \text{diag}(J_1, \ldots, J_m)$
$$J_j = \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_j & 1 \\ & & & \lambda_j \end{bmatrix}$$
So we just need to find the chains of C_ϕ with eigenvalue λ_j.

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2. **Theorem (C&M)**

$\exists$ $H(\mathbb{B}^n)$ a Hilbert space of analytic functions on which C_ϕ compact.

Idea: $C_\phi = \begin{bmatrix} U & 0 \\ V & W \end{bmatrix}$, U is $N \times N$ controls C_ϕ.

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3. U has a λ_j-chain $\Leftrightarrow$ C_ϕ does.
4. So we just find the chains of U, an $N \times N$ matrix.

Example 1

$$\phi(z_1, z_2) = (z_1/2, z_2/4)$$

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$$\phi(z_1, z_2) = (z_1/2, z_2/4) \qquad U = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$$

$$(\phi'(0)^t - 1/2) : z_1 \mapsto 0$$
$$(\phi'(0)^t - 1/4) : z_2 \mapsto 0$$

$$(U - 1/2) : z_1 \mapsto 0$$
$$(U - 1/4) : z_2 \mapsto 0$$
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So Schroeder's Equation has two solutions,

$$F_1 = \begin{bmatrix} c_1 z_1 \\ c_2 z_2 \end{bmatrix}, F_2 = \begin{bmatrix} c_1 z_1 \\ c_2 z_1^2 \end{bmatrix}$$

F_1 has full rank near 0.

Example 2

$$\phi(z_1, z_2) = (z_1/2, z_2/4 + z_1^2/2)$$

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$\phi'(0)^t$ in red and U in purple

$$C_\phi = \begin{array}{c} \\ z_1 \\ z_2 \\ z_1^2 \\ z_1 z_2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} \phi_1 & \phi_2 & \phi_1^2 & \phi_1\phi_2 & \dots \end{array} \\ \begin{bmatrix} 1/2 & 0 & 0 & 0 & \dots \\ 0 & 1/4 & 0 & 0 & \dots \\ 0 & 1/2 & 1/4 & 0 & \dots \\ 0 & 0 & 0 & 1/8 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}$$

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$$U = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$$

$$(\phi'(0)^t \; -1/2) : z_1 \mapsto 0$$

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So Schroeder's Equation has one solution,

$$F = \begin{bmatrix} c_1 z_1 \\ c_2 z_1^2 \end{bmatrix}$$

which is not full rank.

Reduce to Jordan form

$\exists$ D an $n \times n$ invertible matrix $D\phi'(0)D^{-1}$ is in Jordan form.

- Replace ϕ with $\psi = D\phi D^{-1}$ (so $\psi'(0)$ is in Jordan form).

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$$\begin{aligned} C_\psi F &= \psi'(0)F \\ \Leftrightarrow FD\phi(D^{-1}w) &= D\phi'(0)D^{-1}F(w) \\ \Leftrightarrow D^{-1}FD(\phi(z)) &= \phi'(0)(D^{-1}FD)(z), z = D^{-1}w \\ \Leftrightarrow C_\phi(D^{-1}FD) &= \phi'(0)(D^{-1}FD) \end{aligned}$$

- So it suffices to solve Schroeder's equation for ψ.

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- Replace ϕ with $\psi = D\phi D^{-1}$ (so $\psi'(0)$ is in Jordan form).
- So it suffices to solve Schroeder's equation for ψ.

The domain of ϕ is $\mathbb{B}^n$, but the domain of ψ is the ellipsoid $D\mathbb{B}^n$.

- Replace $\mathbb{B}^n$ with $D\mathbb{B}^n$ (an ellipsoid).

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- So it suffices to solve Schroeder's equation for ψ.
- Replace $\mathbb{B}^n$ with $D\mathbb{B}^n$ (an ellipsoid).

We need a Hilbert space of analytic functions on $D\mathbb{B}^n$ on which C_ψ is compact.

$$\begin{array}{ccc} H(\mathbb{B}^n) & \xrightarrow{C_{D^{-1}}} & H(D\mathbb{B}^n) \\ {\scriptstyle C_\phi}\downarrow & & \downarrow{\scriptstyle C_{D\phi D^{-1}}} \\ H(\mathbb{B}^n) & \xrightarrow{C_{D^{-1}}} & H(D\mathbb{B}^n) \end{array}$$

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$$\begin{array}{ccc} H(\mathbb{B}^n) & \xrightarrow{C_{D^{-1}}} & H(D\mathbb{B}^n) \\ C_\phi \downarrow & & \downarrow C_{D\phi D^{-1}} \\ H(\mathbb{B}^n) & \xrightarrow{C_{D^{-1}}} & H(D\mathbb{B}^n) \end{array}$$

Define $H(D\mathbb{B}^n) := C_{D^{-1}} H(\mathbb{B}^n)$, and $\langle f \circ D^{-1}, g \circ D^{-1} \rangle_{H(D\mathbb{B}^n)} = \langle f, g \rangle_{H(\mathbb{B}^n)}$

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- Replace $H(\mathbb{B}^n)$ with $H(D\mathbb{B}^n)$ (so C_ψ is compact).

Was this WLOG?

Yes, it is WITH loss of generality!

$\{z^\alpha\}$ is an orthogonal basis for the original space $H(\mathbb{B}^n)$.

$\Rightarrow \{(Dz)^\alpha\}$ is an orthonormal basis for $H(D\mathbb{B}^n)$.
Very Inconvenient :/

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$\Rightarrow \{(Dz)^\alpha\}$ is an orthonormal basis for $H(D\mathbb{B}^n)$.
In general, $g \in \mathcal{O}(D\mathbb{B}^n)$ is conveniently represented

$$g(z) = \sum a_\alpha z^\alpha$$

- $\{z^\alpha\}$ is a basis for $H(D\mathbb{B}^n)$
- $z^\alpha \perp z^\beta$ is $|\alpha| \neq |\beta|$.
- For example, $z_1 \not\perp z_2$ is possible.

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- For example, $z_1 \not\perp z_2$ is possible.

$$g = \sum a_\alpha z^\alpha = \sum_{s=0}^{\infty} g_s(z)$$

The sum on the right converges in $H(D\mathbb{B}^n)$ since

$$g_s(\lambda z) = \lambda^s g_s(z)$$
$$\Leftrightarrow g_s D(\lambda z) = \lambda^s g_s D$$

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$$J_j = \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_j & 1 \\ & & & \lambda_j \end{bmatrix}$$
So we just need to find the chains of C_ϕ with eigenvalue λ_j.
2. $\exists$ H a Hilbert space of analytic functions on which C_ϕ compact.
3. Idea: $C_\phi = \begin{bmatrix} U & 0 \\ V & W \end{bmatrix}$, U is $N \times N$ controls C_ϕ.
U has a λ_j-chain $\Leftrightarrow$ C_ϕ does.
4. So we just find the chains of U, an $N \times N$ matrix.

Lemma

Let $\lambda \in \mathbb{C} \setminus \{0\}$. $H = H_1 \oplus H_2$ *with* $H_1 \cong \mathbb{C}^N$, $C_\phi = \begin{bmatrix} U & 0 \\ V & W \end{bmatrix}$ *and* $\lambda \notin \sigma(W)$.

$\|W\| = \|(I - P_N)C_\phi(I - P_N)\| < |\lambda|$ for N large, and P_N the projection to the first N coordinates.

$\Rightarrow W - \lambda$ is bounded below, and hence 1-1.

$\Rightarrow W^* - \bar{\lambda}$ is 1-1.

$\Rightarrow W - \lambda$ is invertible, since $H_2 = \ker(W^* - \bar{\lambda}) \oplus (W - \lambda)(H_2)$.

Lemma

If $h = \begin{bmatrix} h_1 \\ \vdots \\ h_N \end{bmatrix} \in \ker(U - \lambda)$, *then* $\exists! \; \tilde{h} = \begin{bmatrix} h_{N+1} \\ \vdots \end{bmatrix} \in H_2$ *such that* $\begin{bmatrix} h \\ \tilde{h} \end{bmatrix} \in \ker(C_\phi - \lambda)$.

Recall, $C_\phi = \begin{bmatrix} U & 0 \\ V & W \end{bmatrix}$, so $-Vh$ is fixed in H_2.

$\Rightarrow \exists! \; \tilde{h} \in H_2$ so that $W\tilde{h} + Vh = 0$

$\Rightarrow (C_\phi - \lambda) \begin{bmatrix} h \\ \tilde{h} \end{bmatrix} = 0.$

Corollary

$\sigma(C_\phi) \setminus \{0\} = diag(C_\phi).$

Chain Correspondence

Theorem

Let $(U-\lambda): E_k \mapsto \ldots \mapsto E_1 \mapsto 0$, *then* $\exists!\ f_j = \begin{bmatrix} E_j \\ \tilde{E}_j \end{bmatrix}$ *so that* $(C_\phi - \lambda): f_k \mapsto \ldots \mapsto f_1 \mapsto 0$.

$k = 1$: Given by the previous lemma.
$k \geq 2$: Notice $(U-\lambda)(U-\lambda)E_2 = 0$, and $(U-\lambda)^2$ is the upper left corner of

$$(C_\phi - \lambda)^2 = C_\phi^2 - 2\lambda C_\phi + \lambda^2 = K + \lambda^2$$

K is lower triangular, and compact, so $\exists!\ \tilde{E}_2$ such that $f_2 = \begin{bmatrix} E_2 \\ \tilde{E}_2 \end{bmatrix} \in \ker[(C_\phi - \lambda)^2]$. Lastly,
$(C-\lambda)f_2 = \begin{bmatrix} E_1 \\ ? \end{bmatrix} \in \ker(C_\phi - \lambda)$. So $? = \tilde{E}_1$ by uniqueness.

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   $$J_j = \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_j & 1 \\ & & & \lambda_j \end{bmatrix}$$ So we just need to find the chains of C_ϕ with eigenvalue λ_j.
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   U has a λ_j-chain $\Leftrightarrow$ C_ϕ does.
4. So we just find the chains of U, an $N \times N$ matrix.

In general $U = \begin{bmatrix} \phi'(0)^t & 0 \\ B & C \end{bmatrix}$ and $\phi'(0)^t = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{bmatrix}$ with each J_j a Jordan block.

In general

$$U = \left[\begin{array}{cccc|ccc|cc}
\lambda & & & & & & & & \\
1 & \lambda & & & & & & & \\
 & \ddots & \ddots & & & & & & \\
 & & 1 & \lambda & & & & & \\
 & & & & J_2 & & & & \\
 & & & & & \ddots & & & \\
 & & & & & & J_m & & \\
\hline
a_1 & a_2 & \dots & a_k & \dots & \dots & \dots a_n & a_{n+1} & \\
 & ? & & & & ? & & ? & \ddots
\end{array}\right]$$

Our task is to put this into Jordan form.

In general

$$U = \left[\begin{array}{cccc|ccc|cc}
\lambda & & & & & & & & \\
1 & \lambda & & & & & & & \\
& \ddots & \ddots & & & & & & \\
& & 1 & \lambda & & & & & \\
& & & & J_2 & & & & \\
& & & & & \ddots & & & \\
& & & & & & J_m & & \\ \hline
a_1 & a_2 & \dots & a_k & \dots & \dots & \dots a_n & a_{n+1} & \\
& ? & & & & ? & & ? & \ddots
\end{array}\right]$$

$$\left[\begin{array}{ccccc|c} \lambda & & & & & 0 \\ 1 & \lambda & & & & \vdots \\ & \ddots & \ddots & & & \vdots \\ & & 1 & \lambda & & 0 \\ \hline a_1 & \dots & a_{k-1} & a_k & & a_{k+1} \end{array}\right]$$

Put into Jordan form.

$$\left[\begin{array}{ccccc|c} \lambda & & & & & 0 \\ 1 & \lambda & & & & \vdots \\ & \ddots & \ddots & & & \vdots \\ & & 1 & \lambda & & 0 \\ \hline a_1 & \dots & a_{k-1} & a_k & & a_{k+1} \end{array}\right]$$

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$$\left[\begin{array}{ccccc|c} \lambda & & & & & 0 \\ 1 & \lambda & & & & \vdots \\ & \ddots & \ddots & & & \vdots \\ & & 1 & \lambda & & 0 \\ \hline 0 & \dots & 0 & 0 & & a_{k+1} \end{array}\right]$$

if $a_{k+1} \neq \lambda$, or

if $a_{k+1} = \lambda$ & $a_k = 0$

$$\left[\begin{array}{ccccc|c} \lambda & & & & & 0 \\ 1 & \lambda & & & & \vdots \\ & \ddots & \ddots & & & \vdots \\ & & 1 & \lambda & & 0 \\ \hline a_1 & \dots & a_{k-1} & a_k & & a_{k+1} \end{array}\right]$$

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if $a_{k+1} = \lambda$ & $a_k \neq 0$

$$
U = \left[\begin{array}{cccc|cccc|cc}
\lambda & & & & & & & & & \\
1 & \lambda & & & & & & & & \\
 & \ddots & \ddots & & & & & & & \\
 & & 1 & \lambda & & & & & & \\
 & & & & J_2 & & & & & \\
 & & & & & \ddots & & & & \\
 & & & & & & J_m & & & \\
\hline
a_1 & a_2 & \ldots & a_k & \ldots & \ldots & \ldots a_n & & a_{n+1} & \\
 & ? & & & & ? & & & ? & \ddots
\end{array}\right]
$$

$$U = \left[\begin{array}{cccc|ccc|cc}
\lambda & & & & & & & & \\
1 & \lambda & & & & & & & \\
 & \ddots & \ddots & & & & & & \\
 & & 1 & \lambda & & & & & \\
 & & & & J_2 & & & & \\
 & & & & & \ddots & & & \\
 & & & & & & J_m & & \\
\hline
0 & 0 & \dots & 0 or 1 & \dots & \dots & \dots 0 or 1 & \lambda & \\
 & ? & & & & ? & & ? & \ddots
\end{array}\right]$$

Theorem (Main Theorem)

- $\exists\, F$ *such that* $C_\phi F = \phi'(0)F$ *with linearly independent component functions.*
- $\exists\, F$ *such that both* $C_\phi F = \phi'(0)F$ *and* $F'(0)$ *full rank* $\Leftrightarrow P\ker(C_\phi - \lambda) = \ker(\phi'(0)^t - \lambda)$ *for each* $\lambda \in \sigma(\phi'(0))$*, with* P *the projection to* $\langle z_1, \ldots z_n \rangle$.

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Corollary

A full rank solution exists
$\Leftrightarrow \dim[P\ker(C_\phi - \lambda)] = \dim[\ker(\phi'(0)^t - \lambda)]$.

Definition

λ_j is a resonant eigenvalue of $\phi'(0)$ iff it occurs on the diagonal of C_ϕ below $\phi'(0)^t$.

Corollary

If there are no resonant eigenvalues of $\phi'(0)$, then a full rank solution exists.

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Corollary

If λ is a resonant eigenvalue and $\dim[\ker(C_\phi - \lambda)] = \dim[\ker(\phi'(0) - \lambda)]$ *no full rank solution can exist.*

Theorem

$\exists\, F_k$ such that $C_\phi F_k = \phi'(0)^k F_k$ with linearly independent component functions for $k = 1, 2, \ldots$. Furthermore, if $k \neq 1$, then F_k cannot have full rank near 0.

We can find F_k explicitly in therms of an F_1.

Algebraic Solutions are Analytic

Definition

λ is a resonant eigenvalue of ϕ if (and only if) $\lambda = \lambda_1^{k_1} \dots \lambda_m^{k_m}$ with λ_j eigenvalues of $\phi'(0)$, and positive integers k_j satisfying $\sum k_j > 2$.

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λ is a resonant eigenvalue of ϕ if (and only if) $\lambda = \lambda_1^{k_1} \ldots \lambda_m^{k_m}$ with λ_j eigenvalues of $\phi'(0)$, and positive integers k_j satisfying $\sum k_j > 2$.

- R. D. Enoch's 2007 paper gives criteria for formal power series solutions to Schroeder's equation.
- Every function from $\mathbb{B}^n$ to $\mathbb{C}$ (for example $\phi_1, \ldots \phi_n$, and $f_1, \ldots, f_n$) is treated as a column vector. Under some hypotheses on ϕ, the coefficients of each f_j are found to satisfy $C_\phi F = \phi'(0)F$ and $\det(F'(0)) \neq 0$.
- Do these formal power series converge on $\mathbb{B}^n$?

Theorem (Bobby & Cowen)

Any formal power series solution to Schroeder's equation is indeed an analytic solution.

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If $\phi'(0)$ has no resonant eigenvalues, then a full rank solution to Schroeder's equation exists.

Theorem (Bobby & Cowen)

If λ is a resonant eigenvalue, and $\dim[\ker(U - \lambda I_N)] = \dim[\ker(\phi'(0)^t - \lambda I_n)]$ then no full rank solution exists.

References

- bridges@purdue.edu - I have a paper that I have not submitted yet, so email me for details.
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