On the Uniqueness of Solutions to the Gross-Pitaevskii Hierarchy with a Switchable Quadratic Trap

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Outline

- Background, physical and mathematical

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- Statement of theorems

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- Conclusion

Background I

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- In lab experiments, the particles are initially confined by traps, e.g., magnetic fields in 1995, then the traps are switched in order to enable observation.
- This is modeled by the N body Schrödinger equation with switchable quadratic potential:

$$i\partial_\tau \psi_N = \frac{1}{2}\left(-\triangle_{\mathbf{y}_N} + \eta(\tau)\left|\mathbf{y}_N\right|^2\right)\psi_N + \sum_{i<j} V_N(y_i - y_j)\psi_N$$

where $\tau \in \mathbb{R}$, $\mathbf{y}_N = (y_1, y_2, ..., y_N) \in \mathbb{R}^{3N}$ and $\eta(\tau)$ is the switch function.

Marginal Density

- Since ψ_N is not a product of one-particle states, the rigorous mathematical description is highly non-trivial. However, the concept of a macroscopic occupation of a single state acquires a precise meaning through the k-particle marginal density $\gamma_N^{(k)}$ associated with ψ_N, where

$$\gamma_N^{(k)}(\tau, \mathbf{y}_k; \mathbf{y}'_k) = \int \psi_N(\tau, \mathbf{y}_k, \mathbf{y}_{N-k}) \overline{\psi_N(\tau, \mathbf{y}'_k, \mathbf{y}_{N-k})} d\mathbf{y}_{N-k}$$

with $\mathbf{y}_k, \mathbf{y}'_k \in \mathbb{R}^{3k}$.

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  with $\mathbf{y}_k, \mathbf{y}'_k \in \mathbb{R}^{3k}$.
- We are here interested in the case $N \to \infty$ in which the Gross-Pitaevskii limit applies. This is also called the mean field approximation.

Gross-Pitaevski hierarchy I

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Gross-Pitaevski hierarchy I

- As $N \to \infty$, consider the equation $\gamma_N^{(k)}$ satisfies. Then we formally arrive at the the switchable quadratic trap Gross-Pitaevskii infinite hierarchy of equations:

$$\left(i\partial_\tau - H_{\mathbf{y}_k}(\tau) + H_{\mathbf{y}'_k}(\tau) \right) \gamma^{(k)} = \sum_{j=1}^{k} B_{j,k+1} \left(\gamma^{(k+1)} \right). \quad (1)$$

where we write $H_{\mathbf{y}_k}(\tau) = \frac{1}{2} \left(-\triangle_{\mathbf{y}_k} + \eta(\tau) \left| \mathbf{y}_k \right|^2 \right)$ for convenience.

Gross-Pitaevski hierarchy I (cont.)

- $B_{j,k+1} = B^1_{j,k+1} - B^2_{j,k+1}$ are defined as

$$\begin{aligned}
& B^1_{j,k+1}\left(\gamma^{(k+1)}\right)(\tau, \mathbf{y}_k; \mathbf{y}'_k) \\
= & \int\int \delta(y_j - y_{k+1})\delta(y_j - y'_{k+1}) \\
& \gamma^{(k+1)}(\tau, \mathbf{y}_{k+1}; \mathbf{y}'_{k+1}) dy_{k+1} dy'_{k+1} \\
& B^2_{j,k+1}\left(\gamma^{(k+1)}\right)(\tau, \mathbf{y}_k; \mathbf{y}'_k) \\
= & \int\int \delta(y'_j - y_{k+1})\delta(y'_j - y'_{k+1}) \\
& \gamma^{(k+1)}(\tau, \mathbf{y}_{k+1}; \mathbf{y}'_{k+1}) dy_{k+1} dy'_{k+1}.
\end{aligned}$$

Why study the G-P Hierarchy?

- Because the cubic NLS

$$i\partial_\tau \phi = H_y(\tau)\phi + |\phi|^2 \phi$$

comes form this linear Gross-Pitaevskii hierarchy. This is why the cubic NLS is also called the Gross-Pitaevskii equation.

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- When we have a condensate initial data

$$\gamma^{(k)}(0, \mathbf{y}_k; \mathbf{y}'_k) = \prod_{j=1}^{k} \phi_0(y_j)\overline{\phi_0(y'_j)}$$

then

$$\gamma^{(k)}(\tau, \mathbf{y}_k; \mathbf{y}'_k) = \prod_{j=1}^{k} \phi(\tau, y_j)\overline{\phi(\tau, y'_j)}$$

where ϕ satisfies the cubic NLS above, is the solution to the Gross-Pitaevskii hierarchy.

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- Away from the solution in the last slide, there is no guarantee that the infinite hierarchy wouldn't have another solution.
- So to justify the mean field approximation, we have to prove the uniqueness of the infinite hierarchy. Our main theorem addresses this.

Previous Work I

- When the switch is off ($\eta = 0$), hierarchy 1 becomes

$$\left(i\partial_t + \frac{1}{2}\triangle_{\mathbf{x}_k} - \frac{1}{2}\triangle_{\mathbf{x}'_k}\right)\gamma^{(k)} = \sum_{j=1}^{k} B_{j,k+1}\left(\gamma^{(k+1)}\right). \qquad (2)$$

which corresponds to the evolution after the removal of the traps. It was studied by Elgart, Erdös, Schlein, and Yau in a series of papers together with the no trapping potential BBGKY hierarchy.

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- Their program consists of two main parts: on the one hand, they prove that an appropriate limit of $\gamma_N^{(k)}$ as $N \to \infty$ solves hierarchy 2, on the other hand, they show that hierarchy 2 has a unique solution.

Previous Work II

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- In 2008, Klainerman and Machedon simplified the proof of uniqueness.
- Later, the method by Klainerman and Machedon was taken up by Kirkpatrick, Schlein, and Staffilani to study the corresponding problem in 2d, and Chen and Pavlović to study the 3-body interaction problem.

Previous Work II

Their key estimate reads: there is $C > 0$, independent of j, k, s.t.

$$\left\| \left(\prod_{j=1}^{k} \left(\nabla_{x_j} \nabla_{x_j'} \right) \right) \left(B^1_{j,k+1} u^{(k+1)} \right) (t, \mathbf{x}_k; \mathbf{x}_k') \right\|_{L^2(\mathbb{R}\times\mathbb{R}^{3k}\times\mathbb{R}^{3k})} \quad (3)$$

$$\leqslant C \left\| \left(\prod_{j=1}^{k+1} \left(\nabla_{x_j} \nabla_{x_j'} \right) \right) u^{(k+1)}(0, \mathbf{x}_{k+1}; \mathbf{x}_{k+1}') \right\|_{L^2(\mathbb{R}^{3(k+1)}\times\mathbb{R}^{3(k+1)})},$$

if $u^{(k+1)}$ verifies non trapping equation

$$\left(i\partial_t + \frac{1}{2}\triangle_{\mathbf{x}_{k+1}} - \frac{1}{2}\triangle_{\mathbf{x}_{k+1}'} \right) u^{(k+1)} = 0.$$

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$$\leqslant \quad C \left\| \left(\prod_{j=1}^{k+1} \left(\nabla_{x_j} \nabla_{x'_j} \right) \right) u^{(k+1)}(0, \mathbf{x}_{k+1}; \mathbf{x}'_{k+1}) \right\|_{L^2(\mathbb{R}^{3(k+1)}\times\mathbb{R}^{3(k+1)})},$$

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- We will also use this estimate in our problem.

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- The 1st condition yields a C^1 even extension of η i.e. we define $\eta(\tau) = \eta(-\tau)$ for $\tau < 0$.
- The fast switching condition (2nd) ensures that the tools to prove our main theorem are well-defined.

The norm we consider

- Define
$$R_\tau^{(k)} = \left(\prod_{j=1}^k P_{y_j}(\tau) P_{y_j'}(-\tau)\right).$$
in which
$$P_y(\tau) = i\beta(\tau)\nabla_y + \dot\beta(\tau) y$$
where β solves
$$\ddot\beta(\tau) + \eta(\tau)\beta(\tau) = 0, \beta(0) = 1, \dot\beta(0) = 0. \tag{5}$$
- We consider the norm $\left\| R_\tau^{(k)} \gamma^{(k)}(\tau, \cdot; \cdot)\right\|_{L^2(\mathbb{R}^{3k}\times\mathbb{R}^{3k})}$

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- β is a non-zero even function in $[-T_0,T_0]$. This is important.
- $P_y(\tau)$ is in fact the evolution of the momentum operator $i\nabla$. It was introduced by Carles in 2010. Himself and Killip-Visan-Zhang used the special case $\eta(\tau)=\pm1$ in many earlier papers.

Uniqueness Theorem

Theorem

Let $\left\{\gamma^{(k)}(\tau,\mathbf{y}_k,\mathbf{y}'_k)\right\}_{k=1}^{\infty}$ *solves the Gross-Pitaevskii hierarchy 1 subject to zero initial data and*

$$\int_0^{T_0}\left\|R_\tau^{(k)}B_{j,k+1}\gamma^{(k+1)}(\tau,\cdot;\cdot)\right\|_{L^2(\mathbb{R}^{3k}\times\mathbb{R}^{3k})}d\tau\leqslant C^k$$

for some $C>0$ *and all* $1\leqslant j\leqslant k$. *Then* $\forall k,\tau\in[0,T_0]$,

$$\left\|R_\tau^{(k)}\gamma^{(k)}(\tau,\cdot;\cdot)\right\|_{L^2(\mathbb{R}^{3k}\times\mathbb{R}^{3k})}=0.$$

Collapsing estimate used to prove the main theorem

Theorem

Let $[s, T] \subset [0, T_0]$. There exists a $C > 0$ independent of j, k, s, and T s.t.

$$\left\| R_\tau^{(k)} B_{j,k+1} \left(\gamma^{(k+1)} \right) \right\|^2_{L^2([s,T]\times\mathbb{R}^{3k}\times\mathbb{R}^{3k})} \leqslant C \left(\sup_{\tau\in[0,T_0]} \frac{1}{(\beta(\tau))^4} \right) \left\| R_\tau^{(k+1)} \gamma^{(k+1)} \right\|^2_{L^2(\mathbb{R}^{3(k+1)}\times\mathbb{R}^{3(k+1)})},$$

where the τ on the RHS of the above estimate can be chosen freely in $[s, T]$, if $\gamma^{(k+1)}(\tau, \mathbf{y}_{k+1}; \mathbf{y}'_{k+1})$ satisfies

$$\begin{aligned} \left(i\partial_\tau - H_{\mathbf{y}_k}(\tau) + H_{\mathbf{y}'_k}(\tau) \right) \gamma^{(k+1)} &= 0 \\ \gamma^{(k+1)}(s, \mathbf{y}_{k+1}; \mathbf{y}'_{k+1}) &= \gamma_s^{(k+1)}(\mathbf{y}_{k+1}; \mathbf{y}'_{k+1}). \end{aligned} \tag{6}$$

A remark on the collapsing estimate

The collapsing estimates can be interpreted as a local smoothing estimate for which integrating in time results in a gain of one hidden derivative in the sense of the trace theorem. Some other collapsing estimates were obtained by myself and Kirkpatrick-Schlein-Staffilani. The work by Yajima and Zhang suggests that the local smoothing effect will be weakened if $|x|^2$ is replaced by $|x|^m$, $m > 2$. Accordingly, $V(x) = |x|^2$ is the strongest possible trap in our setting.

Proof of collapsing estimate

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$$\begin{aligned}
&\int_s^T d\tau \int_{\mathbb{R}^{3k}\times\mathbb{R}^{3k}} \left| R_\tau^{(k)} \gamma^{(k+1)}(\tau, \mathbf{y}_k, y_1; \mathbf{y}'_k, y_1) \right|^2 d\mathbf{y}_k d\mathbf{y}'_k \\
\leqslant\ & C \sup_{\tau\in[0,T_0]} \frac{1}{(\beta(\tau))^4} \int_{\mathbb{R}^{3(k+1)}\times\mathbb{R}^{3(k+1)}} d\mathbf{y}_{k+1} d\mathbf{y}'_{k+1} \\
& \left| R_\tau^{(k+1)} \gamma^{(k+1)}(\tau, \mathbf{y}_{k+1}; \mathbf{y}'_{k+1}) \right|^2,
\end{aligned}$$

$\forall \tau \in [s, T]$, if $\gamma^{(k+1)}$ satisfies equation 6 (with switch).

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$$\leqslant C \sup_{\tau\in[0,T_0]} \frac{1}{(\beta(\tau))^4} \int_{\mathbb{R}^{3(k+1)}\times\mathbb{R}^{3(k+1)}} d\mathbf{y}_{k+1} d\mathbf{y}'_{k+1}$$
$$\left| R_\tau^{(k+1)} \gamma^{(k+1)}(\tau, \mathbf{y}_{k+1}; \mathbf{y}'_{k+1}) \right|^2,$$

$\forall \tau \in [s, T]$, if $\gamma^{(k+1)}$ satisfies equation 6 (with switch). We will need the two lemmas stated below.

Lemma:The generalized lens transform

Lemma

Assume $u^{(k+1)}$ solves equation 4 (no trap) with $u^{(k+1)}(0,\cdot;\cdot)=\gamma_0^{(k+1)}$. Define the generalized lens transform of $u^{(k+1)}$ to be

$$
\begin{aligned}
&Lu^{(k+1)}(\tau,\mathbf{y}_{k+1};\mathbf{y}'_{k+1})\\
=\;&\frac{1}{(\beta(\tau))^{3(k+1)}}u^{(k+1)}\Big(\frac{\alpha(\tau)}{\beta(\tau)},\frac{\mathbf{y}_{k+1}}{\beta(\tau)};\frac{\mathbf{y}'_{k+1}}{\beta(\tau)}\Big)e^{i\frac{\dot{\beta}(\tau)}{\beta(\tau)}\frac{\left(\left|\mathbf{y}_{k+1}\right|^2-\left|\mathbf{y}'_{k+1}\right|^2\right)}{2}},
\end{aligned}
$$

where β is as in equation 5 and

$$\ddot{\alpha}(\tau)+\eta(\tau)\alpha(\tau)=0,\alpha(0)=0,\dot{\alpha}(0)=1.$$

Then in $[0,T_0]$, $Lu^{(k+1)}$ solves equation 6 (with switch).

Lemma:The norm

Lemma

[Carles 2010] $P_y(\tau)$ commutes with the linear operator

$$i\partial_\tau - \frac{1}{2}\left(-\triangle_{\mathbf{y}_k} + \eta(\tau)\left|\mathbf{y}_k\right|^2\right).$$

Moreover,

$$P_y(\tau)U_y(\tau;s)f = U_y(\tau;s)P_y(s)f.$$

where $U_y(\tau;s)$ is the solution operator to

$$(i\partial_\tau - H_y(\tau))u = 0,\ y \in \mathbb{R}^3, u(s,y) = u_s(y)$$

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- Now we can prove the collapsing estimate

Proof I

Use the lens transform we compute

$$R_{\tau}^{(k)}\gamma^{(k+1)}(\tau,\mathbf{y}_k,y_1;\mathbf{y}_k',y_1)$$

$$= \frac{e^{i\frac{\dot{\beta}(\tau)}{\beta(\tau)}\frac{\left(|\mathbf{y}_k|^2-|\mathbf{y}_k'|^2\right)}{2}}}{\left(\beta(\tau)\right)^{k+3}}$$

$$\left(\left(\prod_{j=1}^{k}\left(\nabla_{y_j}\nabla_{y_j'}\right)\right)u^{(k+1)}\left(\frac{\alpha(\tau)}{\beta(\tau)},\frac{\mathbf{y}_k}{\beta(\tau)},\frac{y_1}{\beta(\tau)};\frac{\mathbf{y}_k'}{\beta(\tau)},\frac{y_1}{\beta(\tau)}\right)\right),$$

because

$$i\beta(\tau)\nabla_y\left(e^{i\frac{\dot{\beta}(\tau)}{\beta(\tau)}\frac{\left(|y|^2\right)}{2}}\right)+\dot{\beta}(\tau)y\left(e^{i\frac{\dot{\beta}(\tau)}{\beta(\tau)}\frac{\left(|y|^2\right)}{2}}\right)=0.$$

Proof II

So

$$\int_s^T d\tau \int_{\mathbb{R}^{6k}} \left| R_\tau^{(k)} \gamma^{(k+1)}(\tau, \mathbf{y}_k, y_1; \mathbf{y}'_k, y_1) \right|^2 d\mathbf{y}_k d\mathbf{y}'_k$$

$$\leqslant \left(\sup_{\tau \in [0,T_0]} \frac{1}{(\beta(\tau))^4} \right) \int_{-\infty}^{\infty} dt \int_{\mathbb{R}^{6k}} \left| \left(\prod_{j=1}^{k} \left(\nabla_{x_j} \nabla_{x'_j} \right) \right) u^{(k+1)}(t, \mathbf{x}_k, x_1; \mathbf{x}'_k, x_1) \right|^2 d\mathbf{x}_k d\mathbf{x}'_k$$

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$$\leqslant \left(\sup_{\tau \in [0, T_0]} \frac{1}{(\beta(\tau))^4} \right)$$

$$\int_{-\infty}^{\infty} dt \int_{\mathbb{R}^{6k}} \left| \left(\prod_{j=1}^{k} \left(\nabla_{x_j} \nabla_{x'_j} \right) \right) u^{(k+1)}(t, \mathbf{x}_k, x_1; \mathbf{x}'_k, x_1) \right|^2 d\mathbf{x}_k d\mathbf{x}'_k$$

By estimate 3, we conclude the collapsing estimate.

Key Lemma I

For convenience, we write the solution operator of equation 6 as $U^{(k+1)}(\tau; s)$, then we have lemma:
One can express $\gamma^{(1)}(\tau_1, \cdot; \cdot)$ in the Gross-Pitaevskii hierarchy 1 as a sum of at most 4^n terms of the form

$$\int_D J(\underline{\tau}_{n+1}, \mu_m) d\underline{\tau}_{n+1},$$

or in other words,

$$\gamma^{(1)}(\tau_1, \cdot; \cdot) = \sum_m \int_D J(\underline{\tau}_{n+1}, \mu_m) d\underline{\tau}_{n+1}. \tag{7}$$

Key Lemma II

In last slide, $\underline{\tau}_{n+1} = (\tau_2, \tau_3, ..., \tau_{n+1})$, $D \subset [s, \tau_1]^n$, μ_m are a set of maps from $\{2, ..., n+1\}$ to $\{1, ..., n\}$ satisfying $\mu_m(2) = 1$ and $\mu_m(j) < j$ for all j, and

$$\begin{aligned} J(\underline{\tau}_{n+1}, \mu_m) &= U^{(1)}(\tau_1; \tau_2) B_{1,2} U^{(2)}(\tau_2; \tau_3) B_{\mu_m(3),2} \cdots \\ &\quad U^{(n)}(\tau_n; \tau_{n+1}) B_{\mu_m(n+1),n+1}(\gamma^{(n+1)}(\tau_{n+1}, \cdot; \cdot)). \end{aligned}$$

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In last slide, $\underline{\tau}_{n+1} = (\tau_2, \tau_3, ..., \tau_{n+1})$, $D \subset [s, \tau_1]^n$, μ_m are a set of maps from $\{2, ..., n+1\}$ to $\{1, ..., n\}$ satisfying $\mu_m(2) = 1$ and $\mu_m(j) < j$ for all j, and

$$\begin{aligned} J(\underline{\tau}_{n+1}, \mu_m) &= U^{(1)}(\tau_1; \tau_2) B_{1,2} U^{(2)}(\tau_2; \tau_3) B_{\mu_m(3),2} \cdots \\ & U^{(n)}(\tau_n; \tau_{n+1}) B_{\mu_m(n+1),n+1} (\gamma^{(n+1)}(\tau_{n+1}, \cdot; \cdot)). \end{aligned}$$

- The RHS of formula 7 is in fact a Duhamel principle. This lemma follows from the proof of Theorem 3.4 in [Klainerman-Machedon 2008]. One just needs to replace $e^{i(t_1-t_2)\triangle_y}$ by $U_y(t_1; t_2)$, and $e^{i(t_1-t_2)\triangle^{(k)}}$ by $U^{(k)}(t_1; t_2)$.

The Proof I

Assuming that we have already verified

$$\left\| R_s^{(1)} \gamma^{(1)}(s, \cdot) \right\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} = 0,$$

applying the key lemma to $[s, \tau_1] \subset [0, T_0]$, we have

$$\begin{aligned} & \left\| R_{\tau_1}^{(1)} \gamma^{(1)}(\tau_1, \cdot) \right\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} \\ \leqslant \quad & (\tau_1 - s)^{\frac{1}{2}} \int_{[s,\tau_1]^{n-1}} d\tau_3 ... d\tau_{n+1} \\ & \left\| R_{\tau_2}^{(1)} B_{1,2} U^{(2)}(\tau_2; \tau_3) B_{\mu_m(3),2} \cdots \right\|_{L^2(\tau_2 \epsilon [s,\tau_1] \times \mathbb{R}^3 \times \mathbb{R}^3)} \end{aligned}$$

The Proof II

$$\leqslant \quad C\left(\tau_1 - s\right)^{\frac{1}{2}} \int_{[s,\tau_1]^{n-1}} d\tau_3 ... d\tau_{n+1}$$

$$\left\| R^{(2)}_{\tau_2} U^{(2)}(\tau_2;\tau_3) B_{\mu_m(3),2} ... \right\|_{L^2(\mathbb{R}^6\times\mathbb{R}^6)} \text{ (Collapsing estimate)}$$

$$\leqslant \quad C\left(C\left(\tau_1 - s\right)\right)^{\frac{n-1}{2}}$$

$$\int_s^{\tau_1} \left\| R^{(n)}_{\tau_{n+1}} B_{\mu_m(n+1),n+1} \gamma^{(n+1)}(\tau_{n+1},\cdot) \right\|_{L^2(\mathbb{R}^{3n}\times\mathbb{R}^{3n})} d\tau_{n+1}$$

$$\leqslant \quad C\left(C\left(\tau_1 - s\right)\right)^{\frac{n-1}{2}}.$$

Let $(\tau_1 - s)$ be sufficiently small, and $n \to \infty$, we conclude that

$$\left\| R^{(1)}_{\tau_1} \gamma^{(1)}(\tau_1,\cdot) \right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} = 0 \text{ in } [s,\tau_1].$$

- In this talk, we have established the uniqueness of the hierarchy

$$\left(i\partial_{\tau} - H_{\mathbf{y}_k}(\tau) + H_{\mathbf{y}'_k}(\tau)\right)\gamma^{(k)} = \sum_{j=1}^{k} B_{j,k+1}\left(\gamma^{(k+1)}\right).$$

where $H_{\mathbf{y}_k}(\tau) = \frac{1}{2}\left(-\triangle_{\mathbf{y}_k} + \eta(\tau)\left|\mathbf{y}_k\right|^2\right)$.

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- The next thing to do is to look at the first part of the program, the analysis of the BBGKY hierarchy related to the above hierarchy.
- My talk ends here. Thank you!