[$g(z)$ =Log $f'(z)$]

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Motivation:

f conformal on $\mathbb{D}$

$V = \{g(z) = \text{Log}\, f'(z)\} \subseteq$ Bloch space $= B$

W =interior of the set V in B.

(W is a version of the universal Teichmuller space.)

Theorem 1. *The following holds.*

1) $g(z) = \mathrm{Log}\, f' \in W \Leftrightarrow f$ *has a quasi-conformal extension to* $\mathbb{R}^2 \simeq \mathbb{C}$*.*

2) $g(z) = \mathrm{Log}\, f' \in W \Leftrightarrow \Gamma = f(\partial D)$ *is a quasicircle (e.g.,* $\exists C > 0$*) with* $\forall z, \zeta \in \Gamma$*, diameter* $(\gamma) \leq C|z - \zeta|$ *where* γ *is the shortest subarc of* Γ *containing* z *and* ζ*.*

Let $X \equiv C.T. = (A)^* \simeq (L'/\overline{H_0^1}) \oplus \mathcal{M}_s$. Given $[\mu] \in A^*$, associate

$$f(z) \equiv C * \mu(z) \equiv \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1 - z\bar{\zeta}}.$$
$$\|f\|_X \equiv \inf\{\|\mu\| : f = C * \mu\}.$$

$\exists$ a unique $\nu \in [\mu] \ni \|f\|_X = \|\nu\|$.

Primitives of H^p functions are well studied. For $f \in X$, set

$$If(z) = g(z) \equiv \int_o^z f(\omega) d\omega, \quad |z| < 1.$$

Easy to check that $g \in B$.

Theorem 2. *I maps X into BMOA.*

Proof. Use the integral representation to write (use Fubini)

$$If(z) = -\int_{\mathbb{T}} \zeta \,\mathsf{Log}(1 - t|z|\bar{\zeta})\, d\nu(\zeta)$$

where $z = |z|t$, $|t| = 1$.

Assume p is a polynomial; consider

$$\begin{aligned}\langle g, p\rangle &= \int_{\mathbb{T}} \bar{p}(u)\, g(u)\, dm(u) \\ &= \int_{\mathbb{T}} \bar{p}(u)\Big(-\int_{\pi} \zeta \,\mathsf{Log}(1 - u\bar{\zeta})\, d\nu(\zeta)\Big) dm(u) \\ &= -\int_{\mathbb{T}} \cdot \Big(\int_{\mathbb{T}} \bar{p}(u)\,\mathsf{Log}(1 - u\bar{\zeta}) dm(u)\Big)(-\zeta d\nu(\zeta)).\end{aligned}$$

Hence,

$$|\langle g, p\rangle| \leq \|p\|_{L'}(\|\nu\| \cdot \|\,\mathsf{Log}\,\|_*).$$

Also note: if $G \in H^2 \subset X$ then $\|G\|_X \leq \|G\|_{H^2}$. Further if $f \in X$, $f(z) = \sum_{k=0}^{\infty} \nu_k z^k$ and $\|\nu\| \leq 1$, then

$$If(z) = \sum_{k=0}^{\infty} \frac{\nu_k}{(k+1)} z^{k+1} \qquad \text{and}$$

$$\|If\|_{H^2} = \sqrt{\sum_0^{\infty} \frac{|\nu_k|^2}{(k+1)^2}} \leq \sqrt{\sum_0^{\infty} \frac{1}{(k+1)^2}}.$$

Bourgain has shown that every ball algebra has the Dunford-Pettis property (i.e., every weakly compact bounded linear operator is completely continuous—weakly convergent sequences are mapped to strongly convergent sequences, e.g., $L'(\mu), \mathcal{C}(X))$ and also he showed $A^* \cong X$ has $D + P$ property. $\square$

Proposition 1. *$I : X \to X$ is a compact operator*

Proof. Assume $f_n \in X$, $\|f_n\|_X \leq 1$,

$$g_n(z) = \sum_{k=0}^{\infty} \frac{\nu_k(n)}{(k+1)} z^{k+1} \quad (f = C \times d\nu_n).$$

By the above it is sufficient to prove that some subsequence of g_n converges in H^2. The sequence $\{g_n\}$ is a normal family as well as all of its derivatives. So there is a subsequence of $\{g_n\}$ written as g_j with $g_j(z) \Rightarrow g(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g_j^{(t)}(z) \Rightarrow g^{(t)}(z)$ uniformly on compacts of D. Then $g \in H^2$ and, if

$$g_j(z) = \sum_{k=0} \frac{\nu_k(j) z^{j+1}}{(k+1)},$$

then $\lim_{j \to \infty} \left(\frac{\nu_k(j)}{k+1} \right)$ exists and equals a_k.

Consider then

$$\|g_j - g_p\|_{H^2}^2 = \Big|\sum_{k=0} \frac{|\nu_k(j) - \nu_k(p)|^2)}{(k+1)^2}\Big|.$$

For $\varepsilon > 0 \exists K$ with

$$\Big|\sum_{K=k}^{\infty} \frac{|\nu_k(j) - \nu_k(p)|^2)}{(k+1)^2}\Big| \leq 4 \sum_{K=k}^{\infty} \frac{1}{(k+1)^2} < \varepsilon.$$

Using the sequential converge, we (may) (by choosing more subsequences) have

$$\frac{|\nu_k(j) - \nu_k(p)|^2)}{(k+1)^2} < \frac{\varepsilon}{K}$$

for all $0 \leq k \leq K-1$ with $j, p \geq J$. This proves that some subsequence converges in H^2 and hence in X.

□

Digression. Consider $I : X \to B$.

Proposition 2. $I : \left(L'/\overline{H_0^1}\right) \to B_0$.

Proof. Assume $f \in X$ and $f(z) = C * F(z)$ with $F \in L^1(\mathbb{T})$. Choose $\sigma_n(\zeta, F) = \sigma_n$ as the Cesaro means of F on $\mathbb{T}$ and recall $\sigma_n \to F$ in L^1. Obviously $I\sigma_n = p_n$ is a polynomial and so is in B_0.

Since $\|f - C * \sigma_n\|_X \leq \|F - \sigma_n\|_{L^1} \to 0$ and I is continuous we must have that $\|p_n - If\|_B \to 0$. This is sufficient to conclude that $If \in B_0$. $\square$

Observation. It is known that for $f \in H'$

$$\lim_{r \uparrow 1} f(re^{i\theta})(1-r) = 0 \quad \text{a.e. } \theta$$

$$\text{(and)} \quad |f(z)| \leq \frac{\|f\|_{H'}}{(1-|z|)}.$$

But $H' \subseteq L'/\overline{H_0^1}$ and we have

$$\lim_{|z|\to 1} |f(z)|(1-|z|^2) = 0 \quad \text{uniformly}$$

(e.g., independent of $e^{i\theta}$).

Pommerenke has shown that if g is a Bloch function there is a univalent function h with

$$g(z) = \operatorname{Log} h'.$$

In the case we are concerned with, this can be stated as: for every f in X there is a univalent h with

$$h'(z) = e^{\alpha \int_0^z f(\omega)d\omega}.$$

Using the Jensen's inequality for probability measures and estimates on $|f|$, one can show the following.

Theorem 3. *Assume $F \in H'(X)$ and $IF(z) = \alpha \ln(f')$, with $f \in S$. Then f has a q.c. extension to $\mathbb{C}$ with complex dilation*

$$\mu_f(z) = \frac{1}{\bar{z}^2}\left(z - \frac{1}{\bar{z}}\right) F(z)\left(\frac{1}{\bar{z}}\right), \quad |z| < 1.$$

Also Zinsmeister has defined the following. For Ω a simply connected domain in $\mathbb{C}$, the Carleson measures on Ω are measures ν on Ω for which $\exists\ C > 0$ and $|\nu|(D(z,r) \leq C \cdot r$, $\forall z \in \partial\Omega$ and $r \leq$ diameter $(\partial\Omega)$.

An easy estimate shows that

$$\nu \equiv |\mu|^2(|z|^2 - 1)^{-1} dxdy$$

is a Carleson measure on $\mathbb{C} \backslash D$. In our case the ratio

$$\frac{|\nu|(D(z, r))}{r}$$

has zero limit as $r \to 0$.