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New Criteria for Boundedness and Compactness of Weighted Composition Operators Mapping into the Bloch Space

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Today's Question

Given Banach spaces X and Y and an operator $T : X \to Y$, what is a minimal bounded subset E of X such that

- $$\sup_{x \in E} \|Tx\|_Y < \infty \implies T \text{ is bounded?}$$

- $$\lim_{n\to\infty} \|Tx_n\|_Y = 0, \text{ for each sequence } \{x_n\} \text{ in } E \text{ convergent to 0 pointwise,}$$
  $$\implies T \text{ is compact?}$$

Today's Environment

Notation

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

$$H(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} : f \text{ analytic}\}.$$

The space Y we will be considering

Mostly the Bloch space $\mathcal{B}$ defined as the set of $f \in H(\mathbb{D})$ such that

$$\|f\|_\beta := \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < \infty.$$

$\mathcal{B}$ is a Banach space with norm $\|f\|_{\mathcal{B}} = |f(0)| + \|f\|_\beta$.

Some remarks will be made also for the case of BMOA.

Today's Environment

The spaces X we will be considering

- The Hardy space H^∞ defined as the set of $f \in H(\mathbb{D})$ such that

$$\|f\|_\infty := \sup_{z\in\mathbb{D}} |f(z)| < \infty.$$

- The Bloch space itself.
- The Dirichlet space $\mathcal{D}$ defined as the set of $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{D}} := \left(|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z)\right)^{1/2} < \infty,$$

where A is Lebesgue area normalized by $A(\mathbb{D}) = 1$.

Today's Environment

The spaces X we will be considering

- The Hardy spaces H^p $(1 \leq p < \infty)$ defined as the set of $f \in H(\mathbb{D})$ such that

$$\|f\|_{H^p} := \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p} < \infty.$$

- The weighted Bergman spaces A^p_α $(1 \leq p < \infty,\ \alpha > -1)$ defined as the set of $f \in H(\mathbb{D})$ such that

$$\|f\|_{A^p_\alpha} := \left(\int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha \, dA(z) \right)^{1/p} < \infty.$$

The Hardy spaces can be considered as limiting spaces of A^p_α as α decreases to -1.

The operator we will be considering

Let $\psi, \varphi \in H(\mathbb{D})$, with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, and let X be and Y be Banach spaces of analytic functions defined on $\mathbb{D}$. The weighted composition operator with symbols ψ and φ is the operator $W_{\psi,\varphi}$ defined as

$$W_{\psi,\varphi} f = \psi(f \circ \varphi) = M_\psi C_\varphi f.$$

Motivation

In the study of the Isometry Problem, it has been shown that for many functions spaces, the isometries are weighted composition operators.

- Banach (1932) - $C(K)$, K compact metric space, surjective isometries
- Forelli (1964) - H^p, $p \neq 2$
- Kolaski (1982) - A^p, surjective isometries
- El-Geberly-Wolfe (1985) - disk algebra

Origin of the Problem

Wulan, Zheng and Zhu (2009)

Let $\varphi \in \mathbb{D}$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Then:

$$C_\varphi : \mathcal{B} \to \mathcal{B} \text{ is compact iff } \lim_{n\to\infty} \|\varphi^n\|_\beta = 0.$$

The weighted composition operators mapping into $\mathcal{B}$ have been studied by

- Ohno (2001) - boundedness and compactness for $X = H^\infty$
- Ohno and Zhao (2001) - boundedness and compactness for $X = \mathcal{B}$
- Ohno and Stroethoff (2011) -
  - boundedness and compactness for $X = H^2, A^2_\alpha$
  - boundedness for $X = H^p, A^p_\alpha$, with $1 < p < \infty$, and $\mathcal{D}$

Boundedness of $W_{\psi,\varphi} : X \to \mathcal{B}$

What is Needed?

- $\psi = W_{\psi,\varphi} 1 \in \mathcal{B}$.
- $(1-|z|^2)|(\psi(f \circ \varphi))'(z)| \leq C,$ for $z \in \mathbb{D}$ for $f \in X$, $\|f\|_X = 1$,

or the uniform boundedness of

$$\text{(A)} \qquad (1-|z|^2)|\psi'(z) f(\varphi(z))| \text{ and}$$

$$\text{(B)} \qquad (1-|z|^2)|\psi(z) f'(\varphi(z))\varphi'(z)|.$$

Thus, we need to relate $|f(\varphi(z))|$ and $|f'(\varphi(z))|$ to $\|f\|_X$.

$X = H^\infty$

Then $|f(\varphi(z))| \leq \|f\|_\infty$, so the boundedness of (A) is satisfied just with the assumption that $\psi \in \mathcal{B}$.

Boundedness of $W_{\psi,\varphi} : H^\infty \to \mathcal{B}$

By the Schwarz-Pick Lemma $H^\infty \subseteq \mathcal{B}$ and, for $f \in H^\infty$, $z \in \mathbb{D}$,

$$(1 - |\varphi(z)|^2)|f'(\varphi(z))| \le \|f\|_\infty \text{ so}$$

$$(1 - |z|^2)|\psi(z)\varphi'(z)f'(\varphi(z))| \le \frac{(1 - |z|^2)|\psi(z)\varphi'(z)|}{1 - |\varphi(z)|^2}\|f\|_\infty.$$

Ohno (2001)

(a) $W_{\psi,\varphi}$ is bounded iff $\psi \in \mathcal{B}$ and $\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)|\psi(z)\varphi'(z)|}{1 - |\varphi(z)|^2} < \infty$.

(b) If $W_{\psi,\varphi}$ is bounded, then $W_{\psi,\varphi}$ is compact iff

$$\lim_{|\varphi(z)| \to 1} (1 - |z|^2)|\psi'(z)| = 0 \text{ and}$$

$$\lim_{|\varphi(z)| \to 1} \frac{(1 - |z|^2)|\psi(z)\varphi'(z)|}{1 - |\varphi(z)|^2} = 0.$$

New Characterization of Boundedness (C.)

The following are equivalent:

(a) $W_{\psi,\varphi}$ is bounded.

(b) $\sup_{n\geq 0} \|\psi\varphi^n\|_{\mathcal{B}} < \infty$.

Proof. (a) $\Longrightarrow$ (b): For $n \geq 0$, $z \in \mathbb{D}$, let $p_n(z) = z^n$. Then $p_n \in H^\infty$ and $\|p_n\|_\infty = 1$. Since $W_{\psi,\varphi}$ is bounded,

$$\|\psi\varphi^n\|_{\mathcal{B}} = \|W_{\psi,\varphi} p_n\|_{\mathcal{B}} \leq \|W_{\psi,\varphi}\|.$$

(b) $\Longrightarrow$ (a): By Ohno's Theorem, it suffices to show that

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} < \infty.$$

Let C be an upper bound for $\|\psi\varphi^n\|_{\mathcal{B}}$, $n \geq 0$. Fix $N \geq 2$ and $z \in \mathbb{D}$. For $|\varphi(z)| \leq 1 - \frac{1}{N}$, by the Schwarz-Pick lemma, we have

$$\frac{(1-|z|^2)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} \leq \max\left\{|\psi(w)| : w \in \mathbb{D}, |\varphi(w)| \leq 1 - \frac{1}{N}\right\}. \quad (1)$$

For $|\varphi(z)| > 1 - \frac{1}{N}$, $\exists n > N$ such that $1 - \frac{1}{n-1} \le |\varphi(z)| \le 1 - \frac{1}{n}$. So

$$\frac{(1-|z|^2)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} \le \frac{(1-|z|^2)|\psi(z)n\varphi(z)^{n-1}\varphi'(z)|}{(1-|\varphi(z)|)n|\varphi(z)|^{n-1}}.$$

Let $g(x) = (1-x)nx^{n-1}$ for $1 - \frac{1}{n-1} \le x \le 1 - \frac{1}{n}$. Then

$$g'(x) = nx^{n-2}(n-1-nx) \ge 0.$$

Thus, g is increasing, so its minimum is attained at $1 - \frac{1}{n-1}$ and given by

$$\frac{n}{n-1}\left(1 - \frac{1}{n-1}\right)^{n-1}.$$

As a function of n, this is decreasing and its limit is $1/e$.

Therefore,

$$\begin{aligned}
\frac{(1-|z|^2)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} &\le e(1-|z|^2)|\psi(z)n\varphi(z)^{n-1}\varphi'(z)| \\
&\le e[(1-|z|^2)|(\psi\varphi^n)'(z)| \\
&\qquad +(1-|z|^2)|\psi'(z)\varphi(z)^n|] \\
&\le e(\|\psi\varphi^n\|_{\mathcal{B}} + \|\psi\|_{\mathcal{B}}) \\
&\le 2eC. \qquad (2)
\end{aligned}$$

From (1) and (2), we deduce that

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} < \infty.$$

For $w, z \in \mathbb{D}$, define

$$g_w(z) = \frac{1 - |\varphi(w)|^2}{1 - \overline{\varphi(w)}z}.$$

Theorem (New Characterizations of Compactness, C.)

The following are equivalent:

(a) $W_{\psi,\varphi}$ is compact.

(b) $\lim_{n\to\infty} \|\psi\varphi^n\|_{\mathcal{B}} = 0$ and $\lim_{|\varphi(w)|\to 1} (1 - |w|^2)|\psi'(w)| = 0.$

(c) $\lim_{n\to\infty} \|\psi\varphi^n\|_{\mathcal{B}} = 0$ and $\lim_{|\varphi(w)|\to 1} \|W_{\psi,\varphi} g_w\|_{\mathcal{B}} = 0.$

Ohno-Zhao (2001)

(a) $W_{\psi,\varphi} : \mathcal{B} \to \mathcal{B}$ is bounded iff

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} < \infty \text{ and}$$

$$\sup_{z\in\mathbb{D}} (1-|z|^2)|\psi'(z)| \log\left(\frac{2}{1-|\varphi(z)|^2}\right) < \infty.$$

(b) If $W_{\psi,\varphi}$ is bounded on $\mathcal{B}$, then $W_{\psi,\varphi}$ is compact iff

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} = 0 \quad \text{and}$$

$$\lim_{|\varphi(z)|\to 1} (1-|z|^2)|\psi'(z)| \log\left(\frac{2}{1-|\varphi(z)|^2}\right) = 0.$$

For $w, z \in \mathbb{D}$, with $\varphi(w) \neq 0$, define

$$g_w(z) = \left(\log \frac{1 + \overline{\varphi(w)}z}{1 - \overline{\varphi(w)}z}\right)^2 \left(\log \frac{1 + |\varphi(w)|}{1 - |\varphi(w)|}\right)^{-1}.$$

Then $\|g_w\|_{\mathcal{B}}$ is bounded and h_w approaches 0 as $|\varphi(w)| \to 1$.

Characterization of Boundedness (C.)

The following are equivalent:

(a) $W_{\psi,\varphi}$ is bounded.

(b) $\sup_{w\in\mathbb{D},\varphi(w)\neq 0} \|W_{\psi,\varphi} g_w\|_{\mathcal{B}} < \infty$, and $\sup_{n\geq 0} \|\psi\varphi^n\|_{\mathcal{B}} < \infty$.

Characterization of Compactness (C.)

The following are equivalent:

(a) $W_{\psi,\varphi}$ is compact.

(b) $\lim_{|\varphi(w)|\to 1} \|W_{\psi,\varphi} g_w\|_{\mathcal{B}} = 0$, and $\lim_{n\to\infty} \|\psi\varphi^n\|_{\mathcal{B}} = 0$.

Recall that

$$f \in \mathcal{D} \text{ iff } \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < \infty,$$

where $dA(z) = \frac{1}{\pi} r \, dr \, d\theta$.
$\mathcal{D}$ is a Hilbert space with inner product

$$\langle f, g \rangle = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)} \, dA(z) = a_0 \overline{b_0} + \sum_{n=1}^{\infty} n a_n \overline{b_n},$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$.

Dirichlet kernel

For $z, w \in \mathbb{D}$, define

$$K_z(w) = 1 + \log \frac{1}{1 - \bar{z}w}.$$

Then for $f \in \mathcal{D}$, $z \in \mathbb{D}$, $f(z) = \langle f, K_z \rangle$. Moreover,

$$\|K_z\|_{\mathcal{D}}^2 = 1 + \sum_{n=1}^{\infty} \frac{|z|^{2n}}{n} = 1 + \log \frac{1}{1 - |z|^2}.$$

Thus, by the Cauchy-Schwarz inequality,

$$|f(z)| = |\langle f, K_z \rangle| \leq \|f\|_{\mathcal{D}} \|K_z\|_{\mathcal{D}} = \|f\|_{\mathcal{D}} \left(1 + \log \frac{1}{1 - |z|^2}\right)^{1/2}.$$

Boundedness (Ohno-Stroethoff (2011))

$W_{\psi,\varphi} : \mathcal{D} \to \mathcal{B}$ is bounded iff

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} < \infty \text{ and}$$

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|\psi'(z)| \left(1+\log\frac{1}{1-|\varphi(z)|^2}\right)^{1/2} < \infty.$$

For $w, z \in \mathbb{D}$, define

$$F_w(z) = \frac{1 + K_{\varphi(w)}(z) - \frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z}}{K_{\varphi(w)}(\varphi(w))^{1/2}}.$$

Then $F_w \in \mathcal{D}$ and has bounded Dirichlet norm.

New Characterization of Boundedness (C.)

The following are equivalent:

(a) $W_{\psi,\varphi}$ is bounded.

(b) $\sup_{w\in\mathbb{D}} \|W_{\psi,\varphi}F_w\|_{\mathcal{B}} < \infty$, and $\sup_{n\geq 0} \|\psi\varphi^n\|_{\mathcal{B}} < \infty$.

Compactness (C.)

The following are equivalent:

(a) $W_{\psi,\varphi}$ is compact.

(b) $\lim_{|\varphi(w)|\to 1} \|W_{\psi,\varphi}F_w\|_{\mathcal{B}} = 0$ and $\lim_{n\to\infty} \|\psi\varphi^n\|_{\mathcal{B}} = 0.$

(c) $\lim_{|\varphi(w)|\to 1} \dfrac{(1-|w|^2)|\psi(w)\varphi'(w)|}{1-|\varphi(w)|^2} = 0,$ and

$$\lim_{|\varphi(w)|\to 1} (1-|w|^2)|\psi'(w)| \left(1 + \log \frac{1}{1-|\varphi(w)|^2}\right)^{1/2} = 0.$$

Fix $\alpha > -1$ and $1 < p < \infty$.

Boundedness (Ohno-Stroethoff (2011))

$W_{\psi,\varphi} : A^p_\alpha \to \mathcal{B}$ is bounded iff

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{1+(2+\alpha)/p}} < \infty \quad \text{and}$$

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\psi'(z)|}{(1-|\varphi(z)|^2)^{(2+\alpha)/p}} < \infty.$$

Case $W_{\psi,\varphi} : A^p_\alpha \to \mathcal{B}$

Fix $\alpha > -1$, $1 \le p < \infty$, and $w \in \mathbb{D}$. For $z \in \mathbb{D}$, define the functions

$$
\begin{aligned}
f_w(z) &= \frac{(1-|\varphi(w)|^2)^{1+(2+\alpha)(1-1/p)}}{(1-\overline{\varphi(w)}z)^{3+\alpha}}, \quad \text{and} \\
g_w(z) &= \left(\frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z}\right)^{1/p} f_w(z).
\end{aligned}
$$

New Characterization of Boundedness (C.)

The following are equivalent:

(a) $W_{\psi,\varphi}$ is bounded.

(b) $\sup_{w\in\mathbb{D}} \|W_{\psi,\varphi} f_w\|_{\mathcal{B}} < \infty$, and $\sup_{w\in\mathbb{D}} \|W_{\psi,\varphi} g_w\|_{\mathcal{B}} < \infty$.

Furthermore, Ohno-Stroethoff's theorem holds also for $p = 1$.

Compactness (C.)

The following are equivalent:

(a) $W_{\psi,\varphi}$ is compact.

$$(b) \quad \lim_{|\varphi(w)|\to 1} \|W_{\psi,\varphi} f_w\|_{\mathcal{B}} = 0 \text{ and } \lim_{|\varphi(w)|\to 1} \|W_{\psi,\varphi} g_w\|_{\mathcal{B}} = 0.$$

$$(c) \quad \lim_{|\varphi(w)|\to 1} \frac{(1-|w|^2)|\psi(w)\varphi'(w)|}{(1-|\varphi(w)|^2)^{1+(2+\alpha)/p}} = 0, \text{ and}$$

$$\lim_{|\varphi(w)|\to 1} \frac{(1-|w|^2)|\psi'(w)|}{(1-|\varphi(w)|^2)^{(2+\alpha)/p}} = 0.$$

The above results hold also for the cases $p = 1$ and $\alpha = -1$, yielding characterizations of boundedness and compactness for H^p.

BMOA is the space of $f \in H(\mathbb{D})$ such that

$$\|f\|_* = \sup_{a\in\mathbb{D}} \|f \circ L_a - f(a)\|_{H^2} < \infty,$$

where

$$L_a(z) = \frac{a - z}{1 - \bar{a}z}, \ z \in \mathbb{D}.$$

BMOA is a Banach space with norm

$$\|f\|_{\mathrm{BMOA}} = |f(0)| + \|f\|_*.$$

Boundedness of $W_{\psi,\varphi}$: BMOA $\to$ BMOA (Laitila (2009))

The following are equivalent:

(a) $W_{\psi,\varphi}$ is bounded.

(b) $\sup_{a\in\mathbb{D}} \alpha(\psi,\varphi,a) < \infty$ and $\sup_{a\in\mathbb{D}} \beta(\psi,\varphi,a) < \infty$, where

$$\begin{aligned}\alpha(\psi,\varphi,a) &= |\psi(a)| \| L_{\varphi(a)} \circ \varphi \circ L_a \|_{H^2}, \\ \beta(\psi,\varphi,a) &= \left(\log \frac{2}{1-|\varphi(a)|^2}\right) \|\psi \circ L_a - \psi(a)\|_{H^2}.\end{aligned}$$

Compactness of $W_{\psi,\varphi}$: BMOA $\to$ BMOA (Laitila (2009))

Assume $W_{\psi,\varphi}$ is bounded. The following are equivalent:

(a) $W_{\psi,\varphi}$ is compact.

(b) $\lim_{|\varphi(a)|\to 1} \alpha(\psi,\varphi,a) = 0$, $\lim_{|\varphi(a)|\to 1} \beta(\psi,\varphi,a) = 0$, and

for all $R \in (0,1)$,

$$\lim_{t\to 1} \sup_{|\varphi(a)|\le R} \int_{E(\varphi,a,t)} |(\psi \circ L_a)(\zeta)|^2 \, dm(\zeta) = 0,$$

where $E(\varphi,a,t) = \{\zeta \in \partial\mathbb{D} : |(L_{\varphi(a)} \circ \varphi \circ L_a)(\zeta)| > t\}$ and m is the Lebesgue measure on the circle.

On-Going Research

Boundedness of $W_{\psi,\varphi}$: BMOA $\to$ BMOA (C.)

The following are equivalent:

(a) $W_{\psi,\varphi}$ is bounded.

(b) $\sup_{n\geq 0} \|\psi\varphi^n\|_{\text{BMOA}} < \infty$ and $\sup_{a\in\mathbb{D}} \beta(\psi,\varphi,a) < \infty$.

Compactness of $W_{\psi,\varphi}$: BMOA $\to$ BMOA (C.)

Suppose $W_{\psi,\varphi}$ is bounded. The following are equivalent:

(a) $W_{\psi,\varphi}$ is compact.

(b) $\lim_{n\to\infty} \|\psi\varphi^n\|_{\text{BMOA}} = 0$ and $\lim_{|\varphi(a)|\to 1} \beta(\psi,\varphi,a) = 0$.

Future Research

- Find countable collections for the cases of the Hardy and the Bergman spaces.
- Study the problem for weighted composition operators mapping into other function spaces.