Recent results on superoptimal approximation by meromorphic functions

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1. $\mathbb{D}$ is the open unit disk & $\mathbb{T}$ is the unit circle.

2. $\mathcal{M}_n$ denotes the space of $n \times n$ matrices equipped with the operator norm $\| \cdot \|_{\mathcal{M}_n}$.

3. For an operator $T$ and $k \geq 0$, we define

$$s_k(T) = \inf \{ \| T - R \| : \text{rank} \, R \leq k \}$$

and

$$\| T \|_e = \inf \{ \| T - K \| : K \text{ is compact} \}.$$ 

4. $L^\infty(\mathcal{M}_n)$ is equipped with $\| \Phi \|_\infty = \text{ess sup}_{\zeta \in \mathbb{T}} \| \Phi(\zeta) \|_{\mathcal{M}_n}$. 

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    \| \Phi \|_\infty = \text{ess sup } \| \Phi(\zeta) \|_{\mathbb{M}_n}, \quad \zeta \in \mathbb{T}.
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A finite **Blaschke-Potapov product** of degree $k$ is an $n \times n$ matrix-valued function of the form

$$B(z) = U_0 \left( \begin{array}{cc} \frac{z-a_1}{1-\overline{a_1}z} & \mathbf{0} \\ \mathbf{0} & I_{n-1} \end{array} \right) U_1 \ldots U_{k-1} \left( \begin{array}{cc} \frac{z-a_k}{1-\overline{a_k}z} & \mathbf{0} \\ \mathbf{0} & I_{n-1} \end{array} \right) U_k,$$

where $a_1, \ldots, a_k \in \mathbb{D}$ and $U_0, U_1, \ldots, U_k$ are constant $n \times n$ unitary matrices.

A matrix-valued function $Q \in L^\infty(\mathbb{M}_n)$ is said to have **at most** $k$ **poles** in $\mathbb{D}$ if there is a Blaschke-Potapov product $B$ of degree $k$ such that $QB \in H^\infty(\mathbb{M}_n)$.

$H^\infty_{(k)}(\mathbb{M}_n)$ consists of matrix-valued functions $Q$ with at most $k$ poles in $\mathbb{D}$.
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Nehari-Takagi problem

**Definition**

Let $k \geq 0$. Given $\Phi \in L^\infty(\mathbb{M}_n)$, we say that $Q$ is a **best approximation** in $H^\infty_{(k)}(\mathbb{M}_n)$ to $\Phi$ if $Q$ has at most $k$ poles and

$$\|\Phi - Q\|_{L^\infty(\mathbb{M}_n)} = \text{dist}_{L^\infty(\mathbb{M}_n)}(\Phi, H^\infty_{(k)}(\mathbb{M}_n)).$$

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Superoptimal meromorphic approximation in $H^\infty_k(\mathbb{M}_n)$

**Definition (Young)**

Let $k \geq 0$ and $\Phi \in L^\infty(\mathbb{M}_n)$. We say that $Q$ is a superoptimal meromorphic approximant of $\Phi$ in $H^\infty_k(\mathbb{M}_n)$ if $Q$ has at most $k$ poles in $\mathbb{D}$ and minimizes the essential suprema of singular values $s_j((\Phi - Q)(\zeta)), j \geq 0$, with respect to the lexicographic ordering:

$$Q \text{ minimizes } \sup_{\zeta \in \mathbb{T}} s_0(\Phi(\zeta) - Q(\zeta)) \text{ on } H^\infty_k(\mathbb{M}_n)$$

then... minimize $\sup_{\zeta \in \mathbb{T}} s_1(\Phi(\zeta) - Q(\zeta))$

then... minimize $\sup_{\zeta \in \mathbb{T}} s_2(\Phi(\zeta) - Q(\zeta))$... and so on.

For $j \geq 0$, the number $t_j^{(k)} \overset{\text{def}}{=} \sup_{\zeta \in \mathbb{T}} s_j(\Phi(\zeta) - Q(\zeta))$ is called the $j$th superoptimal singular value of $\Phi$ of degree $k$. 
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Let \( k \geq 0 \) and \( \Phi \in L^\infty(\mathbb{M}_n) \). We say that \( Q \) is a superoptimal meromorphic approximant of \( \Phi \) in \( H^\infty(\mathbb{M}_n) \) if \( Q \) has at most \( k \) poles in \( \mathbb{D} \) and minimizes the essential suprema of singular values \( s_j((\Phi - Q)(\zeta)), j \geq 0 \), with respect to the lexicographic ordering:

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Let $P_+$ and $P_- := I - P_+$ denote the orthogonal projections from $L^2(\mathbb{C}^n)$ onto $H^2(\mathbb{C}^n)$ and $H^2_-(\mathbb{C}^n) = L^2(\mathbb{C}^n) \ominus H^2(\mathbb{C}^n)$, respectively.

Given $\Phi \in L^\infty(\mathbb{M}_n)$, we define

1. the **Toeplitz operator** $T_\Phi : H^2(\mathbb{C}^n) \to H^2(\mathbb{C}^n)$ by

   $$T_\Phi f = P_+ \Phi f \quad \text{for } f \in H^2(\mathbb{C}^n),$$

   and

2. **Hankel operator** $H_\Phi : H^2(\mathbb{C}^n) \to H^2_-(\mathbb{C}^n)$ by

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Why are these operators useful?

**Theorem (AAK-Treil)**

For $\Phi \in L^\infty(\mathbb{M}_n)$, $s_k(H_\Phi) = \text{dist}_{L^\infty(\mathbb{M}_n)}(\Phi, H^\infty_{(k)}(\mathbb{M}_n))$

$= \min \left\{ \text{ess sup}_{\zeta \in \mathbb{T}} s_0(\Phi(\zeta) - Q(\zeta)) : Q \in H^\infty_{(k)}(\mathbb{M}_n) \right\}.$

How about *uniqueness* of superoptimal approximant?

We say that $\Phi$ is *k-admissible* if $\|H_\Phi\|_e$ is less than the smallest non-zero superoptimal singular value of $\Phi$ of degree $k$.

**Theorem (Peller-Young, Treil)**

If $\Phi$ is $k$-admissible and $s_k(H_\Phi) < s_{k-1}(H_\Phi)$, then $\Phi$ has a unique superoptimal meromorphic approximant in $H^\infty_{(k)}(\mathbb{M}_n)$ and $s_j(\Phi(\zeta) - Q(\zeta)) = t_j^{(k)}$ for a.e. $\zeta \in \mathbb{T}$, $j \geq 0$. 

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The Toeplitz operator with symbol $\Phi - Q$

**Theorem**

*Suppose*

1. $\Phi$ is k-admissible,
2. $s_k(H_{\Phi}) < s_{k-1}(H_{\Phi})$, and
3. $\Phi$ has $n$ non-zero superoptimal singular values of degree $k$.

*Then the Toeplitz operator $T_{\Phi - Q}$ is Fredholm and*

$$\text{ind } T_{\Phi - Q} = \dim \ker T_{\Phi - Q}.$$

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Meromorphic approximation
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Can we compute the index of \( T_{\Phi - Q} \)?

**Question:** \( \text{ind } T_{\Phi - Q} = 2k + \mu \)?

Let \( \Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{z}^5 + \frac{1}{3} \bar{z} & -\frac{1}{3} \bar{z}^2 \\ \bar{z}^4 & \frac{1}{3} \bar{z} \end{pmatrix} \). Then

\[
\begin{align*}
s_0(H_\Phi) &= \frac{\sqrt{10}}{3}, \\
s_1(H_\Phi) &= s_2(H_\Phi) = s_3(H_\Phi) = 1, \\
s_4(H_\Phi) &= \frac{1}{\sqrt{2}}, \quad \text{and} \quad s_5(H_\Phi) = \frac{1}{3},
\end{align*}
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and so \( 2k + \mu = 5 \), where \( \mu \) is the multiplicity of \( s_1(H_\Phi) \).

The superoptimal approximant of \( \Phi \) with at most 1 pole is

\[
Q = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{3} \bar{z} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.
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However, \( \text{ind } T_{\Phi - Q} = \text{dim ker } T_{\Phi - Q} = 4 \) even though \( 2k + \mu = 5 \)!
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The superoptimal approximant of $\Phi$ with at most 1 pole is

$$Q = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} \frac{1}{3} \bar{z} & \emptyset \\ \emptyset & \emptyset \end{array} \right).$$

However, $\text{ind } T_{\Phi - Q} = \dim \ker T_{\Phi - Q} = 4$ even though $2k + \mu = 5$!
Let $B$ and $\Lambda$ be Blaschke-Potapov products such that

$$\ker H_Q = BH^2(\mathbb{C}^n) \text{ and } \ker H_{Qt} = \Lambda H^2(\mathbb{C}^n).$$

Let

$$\mathcal{E} = \{ \xi \in \ker H_Q : \|H\Phi\xi\|_2 = \|(\Phi - Q)\xi\|_2 \}$$

and

$$U = \Lambda^t(\Phi - Q)B.$$

**Theorem**

*If the number of superoptimal singular values of $\Phi$ of degree $k$ equals $n$, then*

1. $\mathcal{E} = B \ker T_U$
2. *the Toeplitz operator $T_U$ is Fredholm and*
3. $\text{ind } T_U = \dim \ker T_U \geq n.$
A special subspace

Let $B$ and $\Lambda$ be Blaschke-Potapov products such that

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If the number of superoptimal singular values of $\Phi$ of degree $k$ equals $n$, then

1. $E = B \ker T_U$
2. the Toeplitz operator $T_U$ is Fredholm and
3. $\text{ind } T_U = \dim \ker T_U \geq n$. 

Alberto A. Condori
Meromorphic approximation
A special subspace

Let $B$ and $\Lambda$ be Blaschke-Potapov products such that

$$\ker H_Q = BH^2(\mathbb{C}^n) \quad \text{and} \quad \ker H_{Q^t} = \Lambda H^2(\mathbb{C}^n).$$

Let

$$E = \{ \xi \in \ker H_Q : \|H_\Phi \xi\|_2 = \|(\Phi - Q)\xi\|_2 \}$$

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Alberto A. Condori
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Let $\mathcal{E} = \{ \xi \in \ker H_Q : \| H_\Phi \xi \|_2 = \| (\Phi - Q) \xi \|_2 \}$. Then the Toeplitz operator $T_{\Phi - Q}$ is Fredholm and has index

$$\text{ind } T_{\Phi - Q} = 2k + \dim \mathcal{E}.$$  

In particular, $\dim \ker T_{\Phi - Q} \geq 2k + n$.

Corollary

If all superoptimal singular values of degree $k$ of $\Phi$ are equal, then

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holds, where $\mu$ denotes the multiplicity of the singular value $s_k(H_\Phi)$. 
The index formula

**Theorem**

Let \( \mathcal{E} = \{ \xi \in \text{ker } H_Q : \| H_{\Phi} \xi \|_2 = \| (\Phi - Q) \xi \|_2 \} \). Then the Toeplitz operator \( T_{\Phi - Q} \) is Fredholm and has index

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Open problem #1

Sharp estimates on the “degree” of $Q$:

**Theorem (Peller-Vasyunin)**

*If $\Phi$ is a rational function $2 \times 2$ with poles off $\mathbb{T}$, then “generically” the best analytic approximant $Q$ to $\varphi$ is a rational function and

$$\deg Q \leq \deg \Phi - 2 \quad \text{unless} \quad \Phi \in H^\infty(\mathbb{M}_2).$$

In general, one has

$$\deg Q \leq 2 \deg \Phi - 3$$

and this inequality is sharp!*

#1. What can be said for matrix-valued functions of arbitrary size?
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Open problem #2 & # 3

#2. How can we verify that a matrix-valued function \( \Phi \in L^{\infty} \) has \( n \) non-zero superoptimal singular values?

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