Recent results on superoptimal approximation by meromorphic functions

Alberto A. Condori

Department of Mathematics
Florida Gulf Coast University
acondori@fgcu.edu

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Notation

1. $\mathbb{D}$ is the open unit disk & $\mathbb{T}$ is the unit circle.
2. $\mathbb{M}_n$ denotes the space of $n \times n$ matrices equipped with the operator norm $\|\cdot\|_{\mathbb{M}_n}$.
3. For an operator T and $k \geq 0$, we define

$$s_k(T) = \inf\{\|T - R\| : \text{rank}\, R \leq k\}$$

and

$$\|T\|_{\mathrm{e}} = \inf\{\|T - K\| : K \text{ is compact }\}.$$

4. $L^\infty(\mathbb{M}_n)$ is equipped with $\|\Phi\|_\infty = \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} \|\Phi(\zeta)\|_{\mathbb{M}_n}$.

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Poles and $H^\infty_{(k)}(\mathbb{M}_n)$

A finite **Blaschke-Potapov product** of degree k is an $n \times n$ matrix-valued function of the form

$$B(z) = U_0 \begin{pmatrix} \frac{z-a_1}{1-\bar{a}_1 z} & \mathbb{O} \\ \mathbb{O} & I_{n-1} \end{pmatrix} U_1 \dots U_{k-1} \begin{pmatrix} \frac{z-a_k}{1-\bar{a}_k z} & \mathbb{O} \\ \mathbb{O} & I_{n-1} \end{pmatrix} U_k,$$

where $a_1, \dots, a_k \in \mathbb{D}$ and $U_0, U_1, \dots, U_k$ are constant $n \times n$ unitary matrices.

A matrix-valued function $Q \in L^\infty(\mathbb{M}_n)$ is said to have **at most** k **poles in** $\mathbb{D}$ if there is a Blaschke-Potapov product B of degree k such that $QB \in H^\infty(\mathbb{M}_n)$.

$H^\infty_{(k)}(\mathbb{M}_n)$ consists of matrix-valued functions Q with at most k poles in $\mathbb{D}$.

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Nehari-Takagi problem

Definition

Let $k \geq 0$. Given $\Phi \in L^\infty(\mathbb{M}_n)$, we say that Q is a **best approximation in $H^\infty_{(k)}(\mathbb{M}_n)$ to Φ** if Q has at most k poles and

$$\|\Phi - Q\|_{L^\infty(\mathbb{M}_n)} = \text{dist}_{L^\infty(\mathbb{M}_n)}(\Phi, H^\infty_{(k)}(\mathbb{M}_n)).$$

How can we define "very best" approximation in order to obtain uniqueness?

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How can we define "very best" approximation in order to obtain uniqueness?

Definition (Young)

Let $k \geq 0$ and $\Phi \in L^\infty(\mathbb{M}_n)$. We say that Q is a superoptimal meromorphic approximant of Φ in $H^\infty_{(k)}(\mathbb{M}_n)$ if Q has at most k poles in $\mathbb{D}$ and minimizes the essential suprema of singular values $s_j\left((\Phi - Q)(\zeta)\right)$, $j \geq 0$, with respect to the *lexicographic* ordering:

$$Q \text{ minimizes } \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} s_0(\Phi(\zeta) - Q(\zeta)) \text{ on } H^\infty_{(k)}(\mathbb{M}_n)$$

$$\text{then... minimize } \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} s_1(\Phi(\zeta) - Q(\zeta))$$

$$\text{then... minimize } \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} s_2(\Phi(\zeta) - Q(\zeta)) \dots \text{ and so on.}$$

For $j \geq 0$, the number $t_j^{(k)} \stackrel{\text{def}}{=} \operatorname{ess\,sup}_{\zeta\in\mathbb{T}} s_j(\Phi(\zeta) - Q(\zeta))$ is called the *jth superoptimal singular value of* Φ *of degree* k.

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Superoptimal meromorphic approximation in $H^\infty_{(k)}(\mathbb{M}_n)$

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Tools: Hankel and Toeplitz operators

Let $\mathbb{P}_+$ and $\mathbb{P}_- := I - \mathbb{P}_+$ denote the orthogonal projections from $L^2(\mathbb{C}^n)$ onto $H^2(\mathbb{C}^n)$ and $H^2_-(\mathbb{C}^n) = L^2(\mathbb{C}^n) \ominus H^2(\mathbb{C}^n)$, respectively.

Given $\Phi \in L^\infty(\mathbb{M}_n)$, we define

1. the **Toeplitz operator** $T_\Phi : H^2(\mathbb{C}^n) \to H^2(\mathbb{C}^n)$ by

$$T_\Phi f = \mathbb{P}_+ \Phi f \quad \text{for } f \in H^2(\mathbb{C}^n),$$

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Why are these operators useful?

Theorem (AAK-Treil)

For $\Phi \in L^\infty(\mathbb{M}_n)$, $s_k(H_\Phi) = \mathrm{dist}_{L^\infty(\mathbb{M}_n)}(\Phi, H^\infty_{(k)}(\mathbb{M}_n))$

$$= \min\left\{ \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} s_0(\Phi(\zeta) - Q(\zeta)) : Q \in H^\infty_{(k)}(\mathbb{M}_n) \right\}.$$

How about *uniqueness* of superoptimal approximant?

We say that Φ is *k-admissible* if $\|H_\Phi\|_e$ is less than the smallest non-zero superoptimal singular value of Φ of degree k.

Theorem (Peller-Young, Treil)

If Φ *is k-admissible and* $s_k(H_\Phi) < s_{k-1}(H_\Phi)$, *then* Φ *has a unique superoptimal meromorphic approximant in* $H^\infty_{(k)}(\mathbb{M}_n)$ *and*

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The Toeplitz operator with symbol $\Phi - Q$

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Suppose

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Then the Toeplitz operator $T_{\Phi-Q}$ *is Fredholm and*

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Can we compute the index of $T_{\Phi-Q}$?

Question: ind $T_{\Phi-Q} = 2k + \mu$?

Let $\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{z}^5 + \frac{1}{3}\bar{z} & -\frac{1}{3}\bar{z}^2 \\ \bar{z}^4 & \frac{1}{3}\bar{z} \end{pmatrix}$. Then

$$s_0(H_\Phi) = \frac{\sqrt{10}}{3},\ s_1(H_\Phi) = s_2(H_\Phi) = s_3(H_\Phi) = 1,$$
$$s_4(H_\Phi) = \frac{1}{\sqrt{2}},\ \text{and}\ s_5(H_\Phi) = \frac{1}{3},$$

and so $2k + \mu = 5$, where μ is the multiplicity of $s_1(H_\Phi)$.

The superoptimal approximant of Φ with at most 1 pole is $Q = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{3}\bar{z} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix}$.

However, ind $T_{\Phi-Q} = \dim\ker T_{\Phi-Q} = 4$ even though $2k + \mu = 5$!

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A special subspace

Let B and Λ be Blaschke-Potapov products such that

$$\ker H_Q = BH^2(\mathbb{C}^n) \text{ and } \ker H_{Q^t} = \Lambda H^2(\mathbb{C}^n).$$

Let

$$\mathcal{E} = \{\xi \in \ker H_Q : \|H_\Phi \xi\|_2 = \|(\Phi - Q)\xi\|_2\}$$

and

$$U = \Lambda^t(\Phi - Q)B.$$

Theorem

If the number of superoptimal singular values of Φ of degree k equals n, then

1. $\mathcal{E} = B \ker T_U$
2. *the Toeplitz operator T_U is Fredholm and*
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Let $\mathcal{E} = \{\xi \in \ker H_Q : \|H_\Phi \xi\|_2 = \|(\Phi - Q)\xi\|_2\}$. *Then the Toeplitz operator* $T_{\Phi - Q}$ *is Fredholm and has index*

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In particular, $\dim \ker T_{\Phi - Q} \geq 2k + n$.

Corollary

If all superoptimal singular values of degree k *of* Φ *are equal, then*

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Open problem #1

Sharp estimates on the "degree" of Q:

Theorem (Peller-Vasyunin)

If Φ is a rational function 2×2 with poles off $\mathbb{T}$, then "generically" the best analytic approximant Q to φ is a rational function and

$$\deg Q \leq \deg \Phi - 2 \quad \textit{unless } \Phi \in H^{\infty}(\mathbb{M}_2).$$

In general, one has

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and this inequality is sharp!

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