TOEPLITZ OPERATORS ON BERGMAN SPACES OF POLYANALYTIC FUNCTIONS

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Let $\mathbb{D} = \{z \in \mathbb{C} \ \mid \ |z| < 1\}$.

For $n \geq 1$, a function h is called n-analytic (or polyanalytic of order n) on $\mathbb{D}$ if it satisfies the generalized Cauchy-Riemann equation

$$\frac{\partial^n h}{\partial \bar{z}^n} = 0 \text{ on } \mathbb{D}.$$

It is clear that h is n-analytic if and only if there are analytic functions $h_0, \ldots, h_{n-1}$ on $\mathbb{D}$ such that

$$h(z) = h_0(z) + h_1(z)\bar{z} + \cdots + h_{n-1}(z)\bar{z}^{n-1},$$

for all $z \in \mathbb{D}$.

Let dA denote the normalized Lebesgue area measure on $\mathbb{D}$. The n-analytic Bergman space $A_n^2(\mathbb{D})$ consists of all n-analytic functions which also satisfy

$$\|h\| = \langle h, h\rangle^{1/2} = \left(\int_{\mathbb{D}} |h(z)|^2 dA(z)\right)^{1/2} < \infty.$$

The space A_1^2 is the familiar Bergman space of the unit disk.

The functional $h \mapsto h(z)$ is bounded on $A_n^2(\mathbb{D})$ for any $z \in \mathbb{D}$. Therefore there is a function K_z in $A_n^2(\mathbb{D})$ such that $h(z) = \langle h, K_z\rangle$ for all $h \in A_n^2(\mathbb{D})$ and $z \in \mathbb{D}$. The function $K_z(w)$ is called the kernel function for $A_n^2(\mathbb{D})$.

One of the first studies of $A_n^2(\mathbb{D})$ appeared in a paper by Koselev in 1977.

He discovered

$$K_z(w) = \frac{n}{(1-w\bar{z})^{2n}} \sum_{j=0}^{n-1} (-1)^j \binom{n}{j+1} \binom{n+j}{n} |1-w\bar{z}|^{2(n-1-j)} |w-z|^{2j}$$

$$= \frac{n|1-w\bar{z}|^{2(n-1)}}{(1-w\bar{z})^{2n}} \sum_{j=0}^{n-1} (-1)^j \binom{n}{j+1} \binom{n+j}{n} |\varphi_z(w)|^{2j}.$$

Then $\|K_z\| = n/(1-|z|^2)$.

The normalized reproducing kernel at z is defined by

$$k_z(w) = \frac{(1-|z|^2)|1-w\bar{z}|^{2(n-1)}}{(1-w\bar{z})^{2n}} \sum_{j=0}^{n-1} (-1)^j \binom{n}{j+1} \binom{n+j}{n} |\varphi_z(w)|^{2j}.$$

Since $A^2_n(\mathbb{D})$ is a closed subspace of $L^2(\mathbb{D}, dA)$, there exists the orthogonal projection $P : L^2(\mathbb{D}, dA) \to A^2_n(\mathbb{D})$. For $f \in L^\infty(\mathbb{D})$ we define the Toeplitz operator $T_f : A^2_n(\mathbb{D}) \to A^2_n(\mathbb{D})$ by

$$T_f h = P(fh), h \in A^2_n(\mathbb{D}).$$

We can express T_f as an integral operator

$$T_f h(z) = \int_{\mathbb{D}} f(w)h(w)\overline{K_z(w)}dA(w)$$

for all $h \in A^2_n(\mathbb{D})$.

This formula can be used to define T_f for $f \in L^1(\mathbb{D})$ as a densely defined operator.

Let Δ denote the Laplacian and let $\tilde{\Delta}$ denote the invariant Laplacian. Then $(\tilde{\Delta}u)(z) = (1 - |z|^2)^2(\Delta u)(z)$ for any twice differentiable function u on $\mathbb{D}$.

For any real number $\alpha \geq 0$, the weighted Berezin transform B_α is defined by

$$B_\alpha u(z) = (\alpha + 1) \int_{\mathbb{D}} u(\varphi_z(w))(1 - |w|^2)^\alpha dA(w)$$

for u in $L^1(\mathbb{D})$ and z in $\mathbb{D}$. Note that B_0 is just the standard unweighted Berezin transform.

Theorem A (Ahern-Flores-Rudin). *If $u \in L^1(\mathbb{D})$, then*

$$\tilde{\Delta}\Big(B_\alpha u\Big) = 4(\alpha + 1)(\alpha + 2)(B_\alpha u - B_{\alpha+1} u).$$

This shows that for such u,

$$B_{\alpha+1} u = \left(1 - \frac{\tilde{\Delta}}{4(\alpha + 1)(\alpha + 2)}\right) B_\alpha u.$$

Theorem A implies that for any integer $k \geq 1$, we have $B_k u = q_k(\tilde{\Delta})(B_0 u)$ for $u \in L^1(\mathbb{D})$, where $q_k(\lambda) = \prod_{j=1}^{k} \left(1 - \frac{\lambda}{4j(j+1)}\right)$.

We now define the Berezin transform of a bounded operator T on $A_n^2(\mathbb{D})$ as

$$B(T)(z) = \langle Tk_z, k_z \rangle \text{ for } z \in \mathbb{D}.$$

For $f \in L^1(\mathbb{D})$ we define the Berezin transform $Bf(z)$ by

$$\begin{aligned} Bf(z) &= \int_{\mathbb{D}} f(w)|k_z(w)|^2 dA(w) \\ &= \int_{\mathbb{D}} f(w) \frac{(1-|z|^2)^2}{|1-w\bar{z}|^4} \Big\{ \sum_{j=0}^{n-1} (-1)^j \times \\ &\quad \binom{n}{j+1}\binom{n+j}{n} |\varphi_z(w)|^{2j} \Big\}^2 dA(w). \end{aligned}$$

Theorem 1. *Let $f \in L^1(\mathbb{D})$ be a non-negative function. Then T_f is compact on $A_n^2(\mathbb{D})$ if and only if $Bf(z) \to 0$ as $|z| \uparrow 1$.*

For the usual Bergman space $A^2_1(\mathbb{D})$, Axler and Zheng showed that for $f \in L^\infty(\mathbb{D})$, T_f is compact if and only if $B_0 f(z) \to 0$ as $|z| \to 1$. Their proof faces many difficulties on $A^2_n(\mathbb{D})$.

For two non-zero functions x and y in $A^2_n(\mathbb{D})$, the rank-one operator $x \otimes y$ is defined by $(x \otimes y)(f) = \langle f, y \rangle x$ for $f \in A^2_n(\mathbb{D})$. For z in $\mathbb{D}$, we have

$$B(x \otimes y)(z) = \frac{(1-|z|^2)^2}{n^2} x(z)\bar{y}(z).$$

The Berezin transform is injective on the space of bounded linear operators on $A^2_1(\mathbb{D})$.

It turns out that this is not the case on $A^2_n(\mathbb{D})$ when $n \geq 2$. In fact, if $f(z) = z$, then f and $\bar{f}$ belong to $A^2_n(\mathbb{D})$ and $B(f \otimes 1) = B(1 \otimes \bar{f})$ but it is clear that $f \otimes 1 \neq 1 \otimes \bar{f}$.

Recall that for $f \in L^1$ and $z \in \mathbb{D}$, the formula for $Bf(z)$ has the form

$$\begin{aligned} Bf(z) &= \int_{\mathbb{D}} f(w) \frac{(1-|z|^2)^2}{|1-w\bar{z}|^4} \left\{ \sum_{j=0}^{n-1} (-1)^j \binom{n}{j+1} \binom{n+j}{n} |\varphi_z(w)|^{2j} \right\}^2 dA(w) \\ &= \int_{\mathbb{D}} f(\varphi_z(\zeta)) \left\{ \sum_{j=0}^{n-1} (-1)^j \binom{n}{j+1} \binom{n+j}{n} |\zeta|^{2j} \right\}^2 dA(\zeta), \end{aligned}$$

by the change of variable $w = \varphi_z(\zeta)$. Let μ be the polynomial of degree $2n-2$ defined by

$$\mu(t) = \left\{ \sum_{j=0}^{n-1} (-1)^j \binom{n}{j+1} \binom{n+j}{n} t^j \right\}^2.$$

Rewriting $\mu(t) = b_0 + b_1(1-t) + \cdots + b_{2n-2}(1-t)^{2n-2}$ we see that Bf can be written as

$$\begin{aligned} Bf &= b_0 B_0(f) + \frac{b_1}{2} B_1(f) + \cdots + \frac{b_{2n-2}}{2n-1} B_{2n-2}(f) \\ &= b_0 B_0(f) + \frac{b_1}{2} q_1(\tilde{\Delta}) B_0(f) + \cdots + \frac{b_{2n-2}}{2n-1} q_{2n-2}(\tilde{\Delta}) B_0(f) \\ &= Q(\tilde{\Delta}) B_0(f), \end{aligned}$$

where $Q(t) = b_0 + \frac{b_1}{2} q_1(t) + \cdots + \frac{b_{2n-2}}{2n-1} q_{2n-2}(t)$ and $q_k(t) = \prod_{j=1}^{k} (1 - \frac{t}{4j(j+1)})$ for $1 \leq k \leq 2n-2$.

The formula $Bf = Q(\tilde{\Delta}) B_0 f$ implies that in order to understand B, we need to understand $Q(\tilde{\Delta})$ and B_0.

For any complex number λ, let X_λ denote the linear space of all twice differentiable functions u on $\mathbb{D}$ such that $\tilde{\Delta} u = \lambda u$.

A theorem by Ahern, Flores and Rudin shows that $X_\lambda \cap L^\infty(\mathbb{D}) \neq \emptyset$ if and only if λ belongs to the set

$$\Omega_\infty = \{\lambda \in \mathbb{C} : 4\text{Re } \lambda + (\text{Im } \lambda)^2 \leq 0\}.$$

This shows that if $u \in C^2(\mathbb{D}) \cap L^\infty(\mathbb{D}), \lambda \notin \Omega_\infty$ and $\tilde{\Delta} u = \lambda u$, then $u = 0$. As a consequence, we obtain

Lemma 2. *Let q be a polynomial of degree $s \geq 0$ whose roots lie outside Ω_∞. Suppose u is a function which is $2s$ times continuously differentiable on $\mathbb{D}$ such that $q(\tilde{\Delta})u = 0$ and $(\tilde{\Delta})^j u$ is bounded for all $1 \leq j \leq s$. Then $u = 0$.*

We now need to study the locations of the roots of the polynomial $Q(t)$. Because of the complexity of Q, we have not fully understood the locations of its roots for all $n \geq 2$.

With the help of Maple, we are able to show that for $2 \leq n \leq 25$, all roots of Q lie outside Ω_∞. When $n = 2$, we have

$$\mu(t) = (2 - 3t)^2 = 1 - 6(1 - t) + 9(1 - t)^2$$

and hence

$$Q(t) = 1 - 3q_1(t) + 3q_2(t) = 1 - \frac{t}{8} + \frac{t^2}{64}.$$

Since the roots of Q are $t = 4 \pm 4\sqrt{-3}$, they lie outside Ω_∞.

The following theorem is a generalization of Ahern's Theorem on the range of Berezin transform.

Theorem B (Rao). *Suppose* u *is a bounded function on* $\mathbb{D}$ *such that*

$$B_0(u) = f_1\bar{g}_1 + \cdots + f_m\bar{g}_m$$

for some positive integer m, *where* $f_1, \ldots, f_m$ *and* $g_1, \ldots, g_m$ *are analytic on* $\mathbb{D}$. *Then* u *is harmonic and we have* $u = f_1\bar{g}_1 + \cdots + f_m\bar{g}_m$.

We use Lemma 2 and Theorem B to prove the following results for the case $n \leq 25$.

Proposition 3. *If u and v are bounded harmonic functions on $\mathbb{D}$ such that $T_u T_v$ is of finite rank, then either u or v is the zero function.*

Proposition 4. *If u and v are non-constant bounded harmonic functions on $\mathbb{D}$ such that $[T_u, T_v] = T_u T_v - T_v T_u$ is of finite rank, then either both u and v are analytic, or both $\bar{u}$ and $\bar{v}$ are analytic, or $u = \alpha v + \beta$ for some complex numbers α and β.*

Proposition 5. *If u and v are bounded harmonic functions on $\mathbb{D}$ such that the semi-commutator $[T_u, T_v) = T_u T_v - T_{uv}$ is of finite rank, then either $\bar{u}$ or v is analytic.*

Question 1. *Is it true that all roots of Q lie outside Ω_∞ for all $n \geq 2$?*

Question 2. *Let f be bounded on $\mathbb{D}$. If $B(T_f)(z) \to 0$ as $|z| \uparrow 1$, does it follow that T_f is compact on $A^2_n(\mathbb{D})$ for $n \geq 2$?*