Noncommutative Analogues of the Fejér-Riesz Theorem

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- Yes (Artin 1927).

Some definitions

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Krivine's Abstract Positivstellensatz

Abstract Positivstellensatz (Krivine):

1. $a > 0$ on $W(H)$ iff there exists $t_1, t_2 \in T(H)$ such that $t_1 a = 1 + t_2$.
2. $a \geq 0$ on $W(H)$ iff there exists $t_1, t_2 \in T(H)$, $n \in \mathbb{N}$ such that $t_1 a = a^{2n} + t_2$.
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- On the other hand, if $-1 \notin T(H)$, then $T(H)$ is a prepositive cone, and so extends to a positive cone P. So for each $h \in H$, $h \in T(H) \subset P$, and so $W(H) \neq \emptyset$.

A Concrete Positivstellensatz

Tarski Transfer Principle: Let $(F,\leq)$ be an ordered field extension of $\mathbb{R}$. Then any finite system H of d-variable polynomial inequalities (with coefficients in $\mathbb{R}$) having a solution in F^d has a solution in $\mathbb{R}^d$.

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Suppose $P \in W(H)$. Note that $P \supset T(H)$. Set $F = \operatorname{Quot}(A/\operatorname{supp} P)$ with the ordering induced by P. Note that $\mathbb{R}$ sits inside of F as constant functions, and the ordering on F restricts to the unique ordering on $\mathbb{R}$. Find a solution inside of F and apply the Tarski Transfer Principle to get $x \in \mathbb{R}^d$ such that $h(x) \geq 0$ for all $h \in H$; that is, $W_{\mathbb{R}}(H) \neq \emptyset$.

A generalization of Artin's result

Argue as before that the following statements are equivalent for $A=\mathbb{R}[x_1,\ldots,x_d]$:

1. $a>0$ on $W_{\mathbb{R}}(H)$ iff there exists $t_1,t_2\in T(H)$ such that $t_1a=1+t_2$.
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Then when $H \subset A$ finite, Artin's result is just a corollary of the concrete Positivstellensatz along with 2. when $H = \{1\}$.

Putinar's Theorem and Schmüdgen's Theorem

Recall:

- *Preordering:* $T(H) = \sum_{h \in \prod H} h \sum A^2$.
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The idea of the proof is to show that $T(H)$ (which is a quadratic module) is archimedean. This is done by using the concrete Positivstellensatz and the solution of a moment problem to prove that positive unital linear functionals (relative to $T(H)$) are norm continuous and so in the closed convex hull of states; ie, point evaluations at points in $W_{\mathbb{R}}(H)$.

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- $T(H)$, $Q(H)$ are prepositive cones iff they do not contain -1.
- A prepositive cone T is *archimedean* 1 is in the algebraic interior of T.

Putinar's Theorem: Let $A = \mathbb{R}[x_1, \ldots, x_d]$, $H \subset A$ finite. Suppose that $Q(H)$ is archimedean. If $f > 0$ on $W_{\mathbb{R}}(H)$, then $f \in Q(H)$.

The proof uses a separating hyperplane (Hahn-Banach) argument.

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The idea of the proof is to show that $T(H)$ (which is a quadratic module) is archimedean. This is done by using the concrete Positivstellensatz and the solution of a moment problem to prove that positive unital linear functionals (relative to $T(H)$) are norm continuous and so in the closed convex hull of states; ie, point evaluations at points in $W_{\mathbb{R}}(H)$.

Note that even if $T(H)$ is archimedean, $Q(H)$ need not be.

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In the single variable case, a more careful analysis can be used to get the outer factorization. Likewise in several variables, depending on how you define outer!

Another view

- The complex scalar valued trigonometric polynomials in d variables form a unital involutive algebra $\mathscr{P}$, the involution taking z^n to z^{-n}, where for $n=(n_1,\ldots,n_d)$, $-n=(-n_1,\ldots,-n_d)$. This is the (algebraic) group algebra for $G=\mathbb{Z}^d$.

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- Involution preserving representations of G are unitary, and since the group is commutative, the irreducible representations are 1-dimensional. Group representations of locally compact groups extend naturally to the algebraic group algebra $\mathscr{P}$.

Another view, continued

Alternate definition: A (commutative) multivariable trigonometric polynomial P is positive / strictly positive if for each (topologically) irreducible (and hence 1-dimensional) unitary representation π of G, the extension of π to a unital $*$-representation of the algebra $\mathscr{P} \otimes \mathscr{C}$, has the property that $\pi(P) \geq 0$ / $\pi(P) > 0$.

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Since $\mathbb{T}^d$ is compact, $\pi(P) > 0$ implies the existence of some $\epsilon > 0$ such that $\pi(P - \epsilon 1 \otimes 1) = \pi(P) - \epsilon 1 \geq 0$.

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- This (and what follows) also works if G is an inverse semigroup.

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- We consider the problem of factorizing nonnegative / positive (real) trigonometric polynomials.

McCullough's Theorem

McCullough's noncommutative Fejér-Riesz theorem: Let G be the free group in d noncommuting letters, P a positive trigonometric polynomial of degree n over G with coefficients in $\mathscr{L}(\mathscr{G})$. Then there is a finite collection of analytic polynomials $\{Q_1, \ldots, Q_m\}$ of degree n or less such that $P = \sum_{k=1}^{m} Q_k^* Q_k$.

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What if we only consider strict positivity?

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- For $A, B \in \mathscr{L}(\mathscr{H})$ and $w_1, w_2 \in S$,

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The cone

- Define a cone C in H as the set of nonnegative linear combinations of "squares" (sums of squares) of analytic polynomials.
- Elements of H are sums of terms of the form $1 \otimes A$ or $w_2^* w_1 \otimes B + w_1^* w_2 \otimes B^*$, where $w_1, w_2 \in S$ and A is selfadjoint.
- $1 \otimes A$ is obviously the difference of squares. Since $w^* w = 1$ for any $w \in G$, we also have

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- Iterating shows that C is archimedean: for any $P \in H$, there is some constant $0 \leq \alpha < \infty$ such that $\alpha 1 \pm P \in C$.

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- Much as in the proof of Putinar's theorem, here we used the archimedean property to get the sum of squares decomposition.
- Are there noncommutative analogues of Schmüdgen's theorem? All known proofs of Schmüdgen's theorem ultimately depend on the Tarski Transfer Principle. Is there a noncommutative analogue of this?

The End